EXTERIOR REGULARIZED HARMONIC AND HARMONIC FUNCTIONS

A Dissertation

Presented to

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In Partial Fulfillment

of the Requirements for the Degree

Doctor of Philosophy

By

Qi Han

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I dedicate the thesis to my parents and my dear wife Jingbo Liu.

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Abstract

In this work, we mainly study harmonic functions on an exterior region $U \subseteq \mathbb{R}^N$ ($N \geq 3$), with a compact, Lipschitz boundary ∂U, in our new finite energy function space $E^1(U)$ ($\supsetneq H^1(U)$), via the sequence of exterior harmonic Steklov eigenvalues and associated eigenfunctions. These eigenfunctions consist of an orthogonal basis for the subspace $\mathscr{H}(U)$ of $E^1(U)$ of all functions harmonic in U.

Actually, our results generalize exactly certain well-known results on *Laplace*'s *spherical harmonics* in mathematical physics, *i.e.*, solutions of

$$-\Delta_\partial Y_n(\theta, \phi) = n(n+1) Y_n(\theta, \phi), \quad \forall\, n = 1, 2, \dots. \tag{1}$$

Here, Δ_∂ is the classical *Laplace-Beltrami operator* on S_1, defined as

$$\Delta_\partial := \frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right) + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\phi^2}, \tag{2}$$

where θ, ϕ are the inclination and azimuth (-angles), respectively.

Through the interior and exterior harmonic Steklov eigenvalues and associated eigenfunctions, the space $H^{\frac{1}{2}}(\partial U, d\sigma)$ and its dual space $H^{-\frac{1}{2}}(\partial U, d\sigma)$ are defined, whenever ∂U is compact and Lipschitz. This may not be the traditional definition, as we only require weaker boundary regularity conditions. Moreover, discrete form inner products on $H^{\frac{1}{2}}(\partial U, d\sigma)$, as well as the induced ones on $H^{-\frac{1}{2}}(\partial U, d\sigma)$, can be described from using the interior and exterior harmonic Steklov eigenvalues and associated eigenfunctions, and these norms are shown equivalent.

In the end, series representations of harmonic functions in $\mathscr{H}(G)$ and $\mathscr{H}(U)$

will be obtained. These representations involve the boundary data, and the interior or exterior harmonic Steklov eigenvalues and associated eigenfunctions, which enable us the study of explicit spectral approximations. As an application, for any $g \in H^{\frac{1}{2}}(\partial U, d\sigma)$, a pair of harmonic functions in $H^1(G)$ and $E^1(U)$ can be found such that they share the same Dirichlet data g; while, for any $h \in H^{-\frac{1}{2}}(\partial U, d\sigma)$, a pair of harmonic functions in $H^1(G)$ and $E^1(U)$ can be found again such that they share the same Neumann or Robin data h. Surprisingly, this differs from the classical single and double layer potential methods, as our harmonic functions are determined precisely in terms of their respective boundary data.

Contents

Chapter 1

Introduction

In mathematics, spherical harmonics are the angular portion of a class of solutions of *Laplace*'s *equation*. Represented in a system of spherical coordinates, *Laplace*'s *spherical harmonics* are a specific set of spherical harmonics that forms an orthogonal system, first introduced by P.S. Laplace in 1782. These functions are important in many theoretical and practical applications, particularly in the computation of atomic orbital electron configurations, the representations of gravitational fields as well as the magnetic fields of planetary bodies and stars, and the characterization of the cosmic microwave background radiation. By using the sequence of *Laplace*'s *spherical harmonics*, together with the radial function $|x|$, two families of harmonic functions, one in the unit ball B_1 and the other in its exterior $A_1 := \mathbb{R}^3 \setminus \overline{B}_1$, are easily derived, and each family provides an orthogonal basis for the corresponding, properly defined harmonic function space either in B_1 or in A_1.

The family of orthogonal harmonic functions on the unit ball B_1 is a subset of the Sobolev space $H^1(B_1)$, which conventionally is the standard function space to find weak solutions for various types of partial differential equations. However, for the corresponding family of orthogonal harmonic functions obtained in the complement A_1 of $\overline{B}_1$, not all of them are in $H^1(A_1)$, an example being the function $|x|^{-1}$. Therefore, in order to use exterior harmonic functions, a new Sobolev function space should be introduced. That is, the finite energy space $E^1(A_1)$ (defined in page 13), which only requires L^2-integrability of the gradients of the functions but not the functions themselves. Thereby, $H^1(A_1) \subsetneq E^1(A_1)$.

More generally, for any exterior region $U \subseteq \mathbb{R}^N$ $(N \geq 3)$ with a compact, Lipschitz boundary ∂U, we will have the finite energy space $E^1(U)$ which contains the function $|x|^{2-N}$ and of which $H^1(U)$ is a proper subspace.

Let $E_0^1(U)$ be the subspace of $E^1(U)$ of functions whose traces on ∂U are zero. A function u in $E^1(U)$ is said to be harmonic, provided

$$\int_U \nabla u \cdot \nabla v \, dx = 0, \quad \forall\, v \in E_0^1(U). \tag{1.1}$$

On the other hand, for the study of harmonic functions that are determined by their respective Dirichlet, Neumann, or Robin boundary data, the usual approaches used involve single and double layer potentials. For such problems, the boundary data is usually required be continuous, and there are many open questions about how harmonic functions are influenced by their boundary data.

After $E^1(U)$ is introduced, we develop a theory of exterior harmonic Steklov eigenproblems and then derive a corresponding sequence of eigenvalues and an associated family of eigenfunctions. The traces of these eigenfunctions characterize the boundary fractional Sobolev space $H^{\frac{1}{2}}(\partial U, d\sigma)$ and its dual space $H^{-\frac{1}{2}}(\partial U, d\sigma)$

with respect to the boundary L^2-inner product. Besides, this family of exterior harmonic Steklov eigenfunctions provides an orthogonal basis for the subspace $\mathscr{H}(U)$ of $E^1(U)$ of all functions harmonic on U, and their traces provide an orthonormal basis for the boundary space $L^2(\partial U, d\sigma)$. This enables the series representations of harmonic functions in $E^1(U)$ via their boundary data, combined with the exterior harmonic Steklov eigenvalues and associated eigenfunctions.

When ∂U is compact and Lipschitz, the *Calderón extension theorem* (see Marti [25, theorem 5.3.1], or McLean [26, theorem a.4]) holds. This will imply that there exists an isomorphism between the harmonic function subspace $\mathscr{H}(U)$ in $E^1(U)$ and the one $\mathscr{H}(G)$ in $H^1(G)$, with $G := \mathbb{R}^N \setminus \overline{U}$. In addition, decomposition of the function space $D^1\left(\mathbb{R}^N\right)$ (see Lieb and Loss [24, sections 8.2 and 8.3]) can be accordingly described, using the spaces $\mathscr{H}(G)$ and $\mathscr{H}(U)$.

Much of the work discussed above is true for regularized harmonic functions, which are defined as solutions of the equation

$$\int_U (\nabla u \cdot \nabla v + uv)\, dx = 0, \quad \forall\, v \in H_0^1(U). \tag{1.2}$$

This is the simpler case as everything works out in $H^1(U)$.

We start with a brief review of some background knowledge, and next give the assumptions, definitions and notations in chapter 2. Then, in chapter 3, we shall introduce the finite energy space $E^1(U)$ and show some of its basic properties, for example, we have $\gamma(H^1(G)) = \gamma(E^1(U))$. After that, we endow $E^1(U)$ with an inner product involving the boundary L^2-term, and prove that it is a real Hilbert function space with respect to this inner product $\langle \cdot, \cdot \rangle_{\partial, U}$.

In chapter 4, we shall describe approximations of the spaces $H^1(U)$ and $E^1(U)$, respectively. As $H^1(U)$ and $E^1(U)$ correspond to different situations yet $H^1(U)$ is easier to handle, we first do the approximation for $H^1(U)$ and later in section 4.2, we do it for $E^1(U)$. Moreover, when the boundary ∂U is nice enough, we can show that the *Gauss-Green theorem* holds on $E^1(U)$.

The study of the regularized harmonic and harmonic Steklov eigenproblems in each truncated finite energy space $E^1(U_n)$ will then be the main part of chapter 5. Here, U_n is some bounded region that exhausts U as $n \to \infty$. The methods used involve standard calculus of variations and convex analysis.

Chapter 6 is devoted to the derivation of the sequences of exterior regularized harmonic and harmonic Steklov eigenvalues and associated families of respective Steklov eigenfunctions either in $H^1(U)$ or in $E^1(U)$. The limiting process depends on a *compact trace theorem*, which says that the trace mapping from H^1-functions on certain bounded regions to their boundary values in $L^2(\partial U, d\sigma)$ is compact. In addition, both sets of the exterior Steklov eigenvalues go to ∞.

On the other hand, in chapter 7, we shall continue the study of these exterior Steklov eigenfunctions and show that the exterior regularized harmonic eigenfunctions provide an orthogonal basis for the null space of the operator $\mathcal{L}_1(u) = u - \Delta u$ in $H^1(U)$ while the exterior harmonic eigenfunctions provide an orthogonal basis for the subspace of $E^1(U)$ of all functions that are harmonic on U, from which we can accordingly decompose $H^1(U)$ and $E^1(U)$, respectively. Moreover, via boundary L^2-normalization, the traces of both families of exterior Steklov eigenfunctions will then provide orthonormal bases for the space $L^2(\partial U, d\sigma)$. In consequence, our results actually generalize exactly certain well-known results on *Laplace*'s *spherical* *harmonics* in dimension 3. We refer the reader to Axler, Bourdon and Ramey [13, chapter 5] for a different type of generalized spherical harmonics.

In chapter 8, classical examples of sequences of both the interior and exterior regularized harmonic and harmonic Steklov eigenvalues and associated families of respective Steklov eigenfunctions in the standard regions B_1 and $A_1 := \mathbb{R}^N \setminus \overline{B}_1$ are provided when $N = 3$. They are described using the *modified spherical Bessel functions* in B_1 and A_1, and *Laplace*'s *spherical harmonics* on S_1.

Chapter 9 focuses on the weak solvability of exterior regularized harmonic and harmonic equations, subject to Dirichlet, Neumann, or Robin boundary conditions, in the function spaces $H^1(U)$ and $E^1(U)$, respectively. In particular, the fractional Sobolev space $H^{\frac{1}{2}}(\partial U, d\sigma)$ and its dual space $H^{-\frac{1}{2}}(\partial U, d\sigma)$ are described. Related boundary solution operators are also characterized. We repeat here that our space $H^{\frac{1}{2}}(\partial U, d\sigma)$ and its dual space $H^{-\frac{1}{2}}(\partial U, d\sigma)$ are defined by using the interior and exterior Steklov eigenvalues and associated eigenfunctions. This is not the original definition, because we only require ∂U be compact and Lipschitz.

Finally, in chapter 10, through the exterior regularized harmonic and harmonic Steklov eigenvalues and eigenfunctions, $H^{\frac{1}{2}}(\partial U, d\sigma)$ can be endowed with discrete form inner products to become a real Hilbert function space. These inner products and those via the interior regularized harmonic and harmonic Steklov eigenvalues and eigenfunctions can be shown all equivalent for the function space $H^{\frac{1}{2}}(\partial U, d\sigma)$. In addition, isomorphisms from $H^{\frac{1}{2}}(\partial U, d\sigma)$ to the interior and exterior regularized harmonic and harmonic function spaces are obtained, respectively.

Chapter 2

Assumptions, Definitions, and Notations

A non-empty, open, connected subset U of $\mathbb{R}^N$ $(N \geq 3)$ is called an *exterior region* when its complement, say, $\overline{G} := \mathbb{R}^N \setminus U$, is a non-empty, compact subset. Without loss of generality, let's assume that $0 \notin U$ and write $r_0 := \sup\{|x| : x \notin U\}$, with $|x| := \sqrt{x_1^2 + x_2^2 + \cdots + x_N^2}$ for $x = (x_1, x_2, \ldots, x_N) \in \mathbb{R}^N$, the usual Euclidean norm. For $r > r_0$, define $U_r := U \cap B_r$. Here, B_r is the open ball of radius r in $\mathbb{R}^N$, centered at the origin. Denote the boundary of a set A by ∂A. Then, we have $\partial U_r = \partial U \,\dot{\cup}\, S_r$, where S_r is the boundary of B_r.

For a compact set $K \subset \mathbb{R}^N$, $C(K)$ denotes the Banach space of all real-valued, continuous functions on K endowed with the maximum norm. In addition, $\Omega \subseteq \mathbb{R}^N$ denotes a non-empty, open, connected subset that can be bounded or unbounded. Then, $C_c^1(\Omega)$, as usual, denotes the set of all real-valued, continuously differentiable functions that have compact support in Ω.

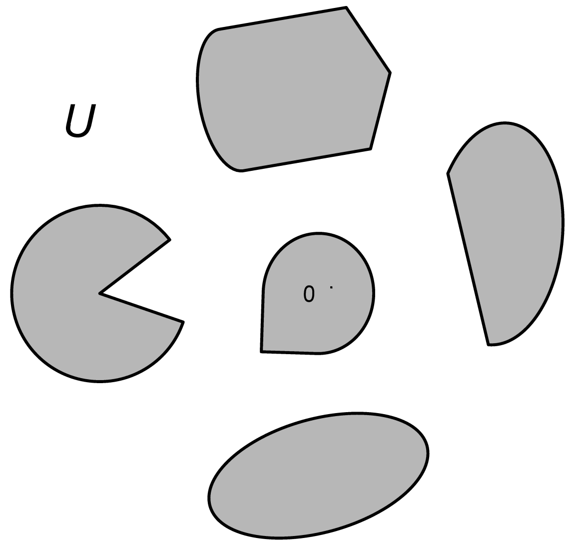

Figure 2.1: Graph of U with the shaded area $G = \mathbb{R}^N \setminus \overline{U}$.

Let $p \in [1, \infty]$, and let all functions be from $\mathbb{R}^N$ to $\overline{\mathbb{R}} := [-\infty, \infty]$. $L^p(\Omega)$ and $L^p(\partial\Omega, d\sigma)$ are the standard spaces of real-valued, Lebesgue measurable functions on Ω and $\partial\Omega$ with their usual norms $||\cdot||_{p,\Omega}$ and $||\cdot||_{p,\partial\Omega}$, respectively. In addition, for the case when $p = 2$, $L^2(\Omega)$ and $L^2(\partial\Omega, d\sigma)$ then are real Hilbert spaces with respect to their respective inner products $\langle\cdot,\cdot\rangle_{2,\Omega}$ and $\langle\cdot,\cdot\rangle_{2,\partial\Omega}$.

$W^{1,p}(\Omega)$ denotes the standard Sobolev space of functions defined on Ω that are in $L^p(\Omega)$ and whose weak derivatives $D_j u$ are again in $L^p(\Omega)$ for $j = 1, 2, \ldots, N$. It is a real Banach space with respect to the usual $W^{1,p}$-norm

$$||u||_{W^{1,p}(\Omega)} := \left(\int_\Omega \left(|u|^p + |\nabla u|^p \right) dx \right)^{\frac{1}{p}}, \tag{2.1}$$

where $\nabla u := (D_1 u, D_2 u, \ldots, D_N u)$ is the weak gradient of u.

In the situation where $p = 2$, we use the notation $H^1(\Omega)$ for $W^{1,2}(\Omega)$. Then, it is a real Hilbert space under the standard H^1-inner product

$$\langle u, v \rangle_{H^1(\Omega)} := \int_\Omega (uv + \nabla u \cdot \nabla v)\, dx, \tag{2.2}$$

and the associated norm is denoted $||u||_{H^1(\Omega)}$.

Given a function $v \in C_c^1(\mathbb{R}^N)$, its restriction to $\overline{\Omega}$ is denoted $R_\Omega(v)$. The set of all such restrictions will be denoted $C_\omega^1(\overline{\Omega})$, and it is a subspace of $W^{1,\infty}(\Omega)$. Let $G^{1,p}(\Omega)$ be the closure of $C_\omega^1(\overline{\Omega})$ with respect to $||\cdot||_{W^{1,p}(\Omega)}$.

Grisvard (see [21, theorem 1.4.2.1]) quotes a result saying $G^{1,p}(\Omega) = W^{1,p}(\Omega)$ when the region Ω has a continuous boundary $\partial\Omega$. DiBenedetto (see [16, chapter vii, propositions 18.1 and 19.1]) shows that $G^{1,p}(\Omega) = W^{1,p}(\Omega)$ when the region Ω satisfies a segment property, and moreover he also provides a counterexample when Ω has a disconnected boundary $\partial\Omega$. Many of the standard extension theorems for C^1-functions on $\overline{\Omega}$ will imply the equality of W and G spaces. See the discussions about extension theorems in Brezis [15, section 9.2], Kufner, John and Fučík [22, chapter 5], and Treves [27, section 26] and its appendix. In fact, whether $W^{1,p}(\Omega)$ equals $G^{1,p}(\Omega)$ or not is a regularity condition on the boundary $\partial\Omega$.

Now, we require some boundary regularity on our U such as

(B1). *U is an exterior region in $\mathbb{R}^N$, with $0 \notin U$, and ∂U is the union of finitely many, disjoint, closed, Lipschitz surfaces, each having finite surface area.*

Remark 2.1. *Note here, under assumption **(B1)**, there exists a bounded extension operator $\mathcal{E} : H^1(U) \to H^1(\mathbb{R}^N)$. Therefore, $H^1(U) = G^{1,2}(U)$, i.e., all functions in $H^1(U)$ can be approximated by sequences of C^1-functions on $\overline{U}$ with respect to*

$||\cdot||_{H^1(U)}$ *that are restrictions of some functions in* $C_c^1\left(\mathbb{R}^N\right)$ *on* $\overline{U}$. *See Marti [25, theorem 5.3.1], and McLean [26, theorem a.4] for more details.*

Remark 2.2. *As a matter of fact, what is mostly needed later for our proofs is to have such extension operators* $\mathcal{E}: H^1(U_r) \to H_0^1(B_{2r})$ *for all* $r > r_0$. *This may only require some weaker regularity conditions on* ∂U *than compact and Lipschitz (see Evans and Gariepy [17, p135, theorem 1] for the latter case). Therefore, we shall call such a* U *a* 1*-extension exterior region when this holds.*

Below, we shall use those definitions and terminology from Evans and Gariepy [17], save that σ and $d\sigma$, respectively, will represent Hausdorff $(N-1)$-dimensional measure and integration with respect to this measure. Besides, Hausdorff $(N-1)$-dimensional measure will be called *surface area measure* in this thesis. In addition, we require that a unique, unit, outward, normal function $\nu : \partial\Omega \to S_1 \subseteq \mathbb{R}^{N-1}$ be defined σ *a.e.* on $\partial\Omega$ for all suitable regions Ω, bounded or not.

A region Ω is said to satisfy the *Rellich-Kondratchov theorem* provided the imbedding $\imath : W^{1,p}(\Omega) \hookrightarrow L^q(\Omega)$ is compact for $1 \leq p < N$ and $1 \leq q < p_S$ with $p_S := \frac{pN}{N-p}$, the *Sobolev conjugate of* p, when $N \geq 3$. Besides, it is said to satisfy the *Sobolev imbedding theorem* provided the mapping $\iota : W^{1,p}(\Omega) \to L^{p_S}(\Omega)$ is also continuous. There exists a number of different criteria on Ω and $\partial\Omega$ implying these results. In particular, when Ω is bounded, and $\partial\Omega$ is compact and Lipschitz, Evans and Gariepy (see [17, p135, theorem 1, p144, theorem 1] plus [18, p279, theorem 2]) give them. Adams and Fournier (see [1, chapters 4 and 6]) treat various conditions for these results thoroughly and show that they are even true for certain classes of unbounded regions that have the property of "small diameters".

Moreover, a region Ω is said to satisfy the *trace theorem* provided the trace mapping $\gamma : W^{1,p}(\Omega) \to L^q(\partial\Omega, d\sigma)$ is continuous for $1 \leq p < N$ and $1 \leq q \leq p_T$ with $p_T := \frac{p(N-1)}{N-p}$ when $N \geq 3$. In addition, it is said to satisfy the *compact trace theorem* provided the trace mapping $\gamma : W^{1,p}(\Omega) \to L^q(\partial\Omega, d\sigma)$ is compact for $1 \leq q < p_T$. The trace operator γ is the linear extension of the mapping restricting *Hölder continuous* functions on $\overline{\Omega}$ to the boundary $\partial\Omega$. When Ω is a bounded region having a compact, Lipschitz boundary $\partial\Omega$, and u is in $W^{1,1}(\Omega)$, the trace of u, denoted γu, then is a well-defined Lebesgue integrable function on $\partial\Omega$ with respect to σ. Evans and Gariepy (see [17, p133, theorem 1]) show that γ is continuous, and Grisvard (see [21, theorem 1.5.1.10]) proves an inequality that implies this compact trace result. Also, DiBenedetto [16, chapter ix, section 18] and Leoni [23, chapter 15] both give results of this type. Recently, Auchmuty [9] derived different trace inequalities with sharp bounds.

On the other hand, when Ω is a region with a compact, Lipschitz boundary $\partial\Omega$, for all $u, v \in H^1(\Omega)$, the *Gauss-Green theorem* holds in the form below

$$\int_\Omega (vD_j u + uD_j v)\, dx = \int_{\partial\Omega} (\gamma u \cdot \gamma v)\, \nu_j \, d\sigma, \tag{2.3}$$

where $\nu_j : \partial\Omega \to \mathbb{R}$ denotes the j-th component of ν for $j = 1, 2, \ldots, N$. Evans and Gariepy (see [17, p209, theorem 1]) show that (2.3) holds for $u, v \in C_c^1(\mathbb{R}^N)$, and the general case follows from the density of $C_c^1(\mathbb{R}^N)$ in $H^1(\mathbb{R}^N)$ and remarks 2.1 and 2.2. By assumption **(B1)** on U, plus remarks 2.1 and 2.2, the *Gauss-Green theorem* holds on the space $H^1(U)$, as now $H^1(U) = G^{1,2}(U)$.

In addition, an identity related to self-adjoint, divergence form partial differential equations, as will be studied in this thesis, can be derived from (2.3). That is, for $u, v \in H^1(\Omega)$ with Δu being in $L^2(\Omega)$, we have

$$\int_\Omega [(\Delta u)\, v + \nabla u \cdot \nabla v]\, dx = \int_{\partial\Omega} (D_\nu u)\, \gamma v\, d\sigma. \tag{2.4}$$

Here, the unit, outward, normal differential operator $D_\nu : H^1(\Omega) \to H^{-\frac{1}{2}}(\partial\Omega, d\sigma)$ is defined as, for all $u \in H^1(\Omega)$, $D_\nu u := \nabla u \cdot \nu$, and the condition $\gamma v \in H^{\frac{1}{2}}(\partial\Omega, d\sigma)$ is applied to justify $\langle D_\nu u, \gamma v\rangle_{2,\partial\Omega}$ (see Auchmuty [4, sections 5-8]).

Below, we give another boundary regularity requirement such as

(B2). *For each $r > r_0$, the bounded region U_r is such that the Rellich-Kondratchov, Sobolev imbedding, compact trace and Gauss-Green theorems hold.*

In the following, we shall apply various standard results from the calculus of variations and convex analysis. Background materials on these methods may be found in Blanchard and Brüning [14], and Zeidler [28], both of which have some discussions for the variational principles concerning the Dirichlet eigenvalues and eigenfunctions of second-order elliptic operators. The variational methods used by us here are variants of those described there and analogous to those employed by Auchmuty [2, 3]. In addition, all variational principles and functionals are defined on some closed, convex subsets of $H^1(U_r)$. A functional $\mathcal{F} : H^1(U_r) \to (-\infty, \infty]$ is said to be $\mathcal{G}$-*differentiable* (*Gâteaux*) at some $u \in H^1(U_r)$, if there is a continuous linear functional $\mathcal{F}'(u)$ on $H^1(U_r)$ such that, for all $v \in H^1(U_r)$,

$$\lim_{t \to 0} \frac{\mathcal{F}(u + tv) - \mathcal{F}(u)}{t} = \mathcal{F}'(u)(v). \tag{2.5}$$

In this situation, $\mathcal{F}'(u)$ is called the $\mathcal{G}$-*derivative* (*Gâteaux*) of $\mathcal{F}$ at u.

Chapter 3

The Finite Energy Space $E^1(U)$

For a bounded region Ω, the standard Sobolev function space $H^1(\Omega)$ is Hilbert with respect to the inner product (2.2). Moreover, when $\partial\Omega$ is compact and Lipschitz, all these functions in $L^1(\Omega)$, with gradients in $L^2(\Omega; \mathbb{R}^N)$, are in $L^2(\Omega)$ by *Poincaré*'s *inequality* (see [17, p144, theorem 1] plus [18, p290, theorem 1]) as

$$\int_\Omega u^2\, dx \le \mathcal{M}(\Omega) \cdot \overline{u}_\Omega^2 + \int_\Omega |\nabla u|^2\, dx, \tag{3.1}$$

where $\overline{u}_\Omega := \frac{1}{\mathcal{M}(\Omega)} \int_\Omega u\, dx$. Here, $\mathcal{M}$ is the usual N-dimensional Lebesgue measure. We thus can conclude that, whenever the estimate (3.1) holds,

$$H^1(\Omega) = \left\{ u : u \in L^1(\Omega) \text{ and } \nabla u \in L^2\left(\Omega; \mathbb{R}^N\right) \right\}. \tag{3.2}$$

When $\mathcal{M}(\Omega)$ is infinite, there is no simple rule for the comparison of different function spaces $L^p(\Omega)$. Also, for our exterior region U, there are functions, such as $f(x) = |x|^{2-N}$, that have gradients in $L^2\left(U; \mathbb{R}^N\right)$ yet it is not in $L^2(U)$. Moreover, we shall require some decay at infinity of the functions to be considered.

We define the *finite energy space* $E^1(U)$ to be the subspace of $L^1_{loc}\left(\overline{U}\right)$ of all functions u such that the following two properties are satisfied.

(A1). u vanishes at infinity in a measure-theoretic sense, that is, for all $a > 0$,

$$\mathcal{M}_u(a) := \mathcal{M}\left(\{x \in U : |u(x)| > a\}\right) < \infty.$$

(A2). The weak gradient ∇u of u is in the space $L^2\left(U; \mathbb{R}^N\right)$.

Notice the term *finite energy* refers to L^2-integrability of the gradients.

From *Chebyshev*'s *inequality* (see Folland [20, theorem 6.17]), one has

$$\mathcal{M}_u(a) \leq a^{-p} \int_{\{x \in U : |u(x)| > a\}} |u|^p \, dx \leq \left(a^{-1} \|u\|_{p,U}\right)^p < \infty \tag{3.3}$$

for all $u \in L^p(U)$, with $1 \leq p < \infty$ and $a > 0$, so that **(A1)** holds. This implies

$$H^1(U) \subsetneq E^1(U). \tag{3.4}$$

These results lead to proofs that the weak solutions of

$$\mu^2 u - \Delta u = 0 \text{ in } U, \text{ subject to } \gamma u = \eta \text{ on } \partial U \tag{3.5}$$

are in $H^1(U)$, where $\mu > 0$ and η is some suitable boundary data. When $\mu = 0$, however, the weak solutions of

$$-\Delta u = 0 \text{ in } U, \text{ subject to } \gamma u = \eta \text{ on } \partial U \tag{3.6}$$

need not be all in $H^1(U)$, such as $u(x) = |x|^{2-N}$ for $|x|$ sufficiently large (see also Auchmuty and Han [10, 11]). In consequence, for the harmonic problems, we must drop the requirement that $u \in L^2(U)$, and use the function space $E^1(U)$ instead.

Let's for the moment go back again to the case when Ω is a bounded region with $\partial\Omega$ compact and Lipschitz. *Sobolev imbedding theorem* (see [17, p135, theorem 1, p138, theorem 1] plus [18, p279, theorem 2]) gives $H^1(\Omega) \subseteq L^{2_S}(\Omega)$ for $2_S := \frac{2N}{N-2}$ (see [1, section 4.3]). Noticing $2 < 2_S$, one actually has that

$$H^1(\Omega) = \left\{u : u \in L^{2_S}(\Omega) \text{ and } \nabla u \in L^2\left(\Omega; \mathbb{R}^N\right)\right\}. \tag{3.7}$$

Fix a $r\,(> r_0 > 0)$, with r_0 being $\sup\{|x| : x \notin U\}$. For the exterior region U, and two given constants r_1, r_2 satisfying $r > r_2 > r_1 > r_0$, define

$$f(|x|) := \begin{cases} 0 & \text{on } B_{r_1} \cup \left\{\mathbb{R}^N \setminus \overline{B}_{r_2}\right\}, \\ \\ \exp\left(\frac{1}{(|x|-r_1)(|x|-r_2)}\right) & \text{on } \overline{B}_{r_2} \setminus B_{r_1}, \end{cases} \tag{3.8}$$

and accordingly write $X(|x|) := \frac{\int_{|x|}^{+\infty} f(t)\,dt}{\int_{-\infty}^{+\infty} f(t)\,dt}$.

Let $\mathcal{X}(x)$ be the restriction of $X(|x|)$ on $\overline{U}$. Then, $\mathcal{X}$ is a decreasing, smooth function, lying in between $[0,1]$, such that

$$\begin{cases} \mathcal{X} \equiv 1 & \text{on } \overline{U}_{r_1}, \\ \\ \mathcal{X} \equiv 0 & \text{on } U \setminus U_{r_2}, \end{cases} \tag{3.9}$$

and such that $||\nabla \mathcal{X}||_{\infty,U} < \infty$.

Given $u \in E^1(U)$, set $\breve{u} := (1-\mathcal{X})\,u$. By zero extension to $\mathbb{R}^N$, $\breve{u} \in L^1_{loc}\left(\mathbb{R}^N\right)$. Applying *Cauchy*'s and *Poincaré*'s *inequalities* (see (3.1)), we have

$$\begin{aligned}
&\int_{\mathbb{R}^N} |\nabla \breve{u}|^2\,dx = \int_{\mathbb{R}^N} |(1-\mathcal{X})\,\nabla u - (\nabla \mathcal{X})\,u|^2\,dx \\
&\leq \int_U |\nabla u|^2\,dx + ||\nabla \mathcal{X}||^2_{\infty,U} \int_{U_r} u^2\,dx + 2\,||\nabla \mathcal{X}||_{\infty,U} \int_{U_r} |u \nabla u|\,dx \\
&\leq 2\left(\int_U |\nabla u|^2\,dx + ||\nabla \mathcal{X}||^2_{\infty,U}\left(\mathcal{M}(U_r)\cdot \overline{u}^2_{U_r} + \int_U |\nabla u|^2\,dx\right)\right) < \infty.
\end{aligned} \tag{3.10}$$

Hence, $\nabla \breve{u} \in L^2\left(\mathbb{R}^N;\mathbb{R}^N\right)$. Also, $\breve{u}$ vanishes at infinity as u does, for G is a bounded region. Thereby, $\breve{u}$ is in $D^1\left(\mathbb{R}^N\right)$, which is the global version of our function space $E^1(U)$ in $\mathbb{R}^N$, that is, $w \in D^1\left(\mathbb{R}^N\right)$ whenever $w \in L^1_{loc}\left(\mathbb{R}^N\right)$, w vanishes at infinity in the sense of **(A1)** for $\mathbb{R}^N$, and $\nabla w \in L^2\left(\mathbb{R}^N;\mathbb{R}^N\right)$. *Sobolev imbedding theorem for gradients* (see Lieb and Loss [24, sections 8.2 and 8.3]) derives $\breve{u} \in L^{2s}\left(\mathbb{R}^N\right)$. In particular, the restriction of $\breve{u}$ on $U \setminus U_{r_2}$, equaling $u|_{U\setminus U_{r_2}}$, is in $L^{2s}(U \setminus U_{r_2})$. Besides, $u|_{\overline{U}_r}$ is in $H^1(U_r)$ by (3.2), so that it is also in $L^{2s}(U_r)$ by (3.7). We thus obtain $u \in L^{2s}(U)$. In consequence, it actually says that

Proposition 3.1. *A function u is in $E^1(U)$ if and only if it is in $L^{2s}(U)$ and its gradient ∇u is in $L^2\left(U;\mathbb{R}^N\right)$.*

As a result, we can derive the following comparison result on different function spaces $L^p(U)$ when $1 \leq p < \infty$, on condition that all these functions involved are such that their weak gradients are in the space $L^2\left(U;\mathbb{R}^N\right)$.

Corollary 3.2. *When **(B1)** holds, for any function u on U with $\nabla u \in L^2\left(U;\mathbb{R}^N\right)$, if u is in $L^p(U)$ for some $p \in [1,\infty)$, then u will be in $L^{2s}(U)$.*

Recall that $G = \mathbb{R}^N \setminus \overline{U}$. Evans and Gariepy (see [17, p135, theorem 1]) give us the existence of a bounded extension operator $\mathcal{E} : H^1(G) \to H_0^1(B_r)$ since $G \Subset B_r$ for $r > r_0$. Restrict the space $H_0^1(B_r)$ on U_r. For all $u \in H_0^1(B_r)$, $u|_{U_r}$ will be in $E^1(U)$ by zero extension to infinity. That is, for all $u \in H^1(G)$, there exists some $\check{u} \in E^1(U)$ such that $\gamma\check{u} = \gamma u$ on ∂U. Besides, for all $u \in E^1(U)$, write $\upsilon := \mathcal{X}u$. Then, $\gamma\upsilon = \gamma u$ on ∂U and $\gamma\upsilon \equiv 0$ on S_{r_2}. Again, we will have a bounded extension operator $\mathcal{E} : H^1(U_{r_2}) \to H_0^1(B_r)$ since $U_{r_2} \Subset B_r$ for $r > r_2$. Restrict $H_0^1(B_r)$ on G. For all $u \in H_0^1(B_r)$, one has $u|_G \in H^1(G)$ as $G \Subset B_r$. Note here, $\upsilon \in H^1(U_{r_2})$ by the boundedness of $\nabla\mathcal{X}$ and (3.2). That is, for all $u \in E^1(U)$, there exists some $\hat{u} \in H^1(G)$ such that $\gamma\hat{u} = \gamma u$ on ∂U. Therefore, we derive that

$$\gamma\left(E^1(U)\right) = \gamma\left(H^1(G)\right) \text{ on } \partial U. \tag{3.11}$$

This argument, plus remarks 2.1 and 2.2, can also show that, on ∂U, the trace spaces of $H^1(U)$ and $H^1(G)$ are the same, which implies that

$$\gamma\left(E^1(U)\right) = \gamma\left(H^1(U)\right) \text{ on } \partial U. \tag{3.12}$$

On the other hand, another type of *Poincaré*'s *inequality* (see [17, p138, theorem 1] plus [18, p279, theorem 3]) says $||u||_{H^1(G)}$ and $||\nabla u||_{L^2(G;\mathbb{R}^N)}$ are equivalent on $H_0^1(G)$, the subspace of $H^1(G)$ of all functions whose traces are zero on ∂G. If we view $H_0^1(G)$ as a subspace of $D^1\left(\mathbb{R}^N\right)$, rather than $H^1\left(\mathbb{R}^N\right)$, since only the gradients matter, we shall get something new, as $H^1\left(\mathbb{R}^N\right) \subsetneq D^1\left(\mathbb{R}^N\right)$. This also explains how we got $E^1(U)$, as now (3.11) and (3.12) lead to

$$H^1(U) \cong H^1\left(\mathbb{R}^N\right)/H_0^1(G), \tag{3.13}$$

with the notation $\cong$ denoting (algebraic) quotient isomorphism, and

$$E^1(U) \cong D^1\left(\mathbb{R}^N\right)/H_0^1(G). \tag{3.14}$$

This can be regarded as the third characterization of $E^1(U)$.

First of all, we show that $D^1\left(\mathbb{R}^N\right)$ is a real vector space. Actually, let $w, w^* \in D^1\left(\mathbb{R}^N\right)$, and let $c \neq 0$ be a constant. For all $a > 0$, we have

$$\begin{cases} \mathcal{M}_{w+w^*}(a) \leq \mathcal{M}_w\left(\frac{a}{2}\right) + \mathcal{M}_{w^*}\left(\frac{a}{2}\right) < \infty, \\ \\ \mathcal{M}_{cw}(a) \leq \mathcal{M}_w\left(\frac{a}{|c|}\right) < \infty. \end{cases} \tag{3.15}$$

Noticing that $\mathcal{M}(G) < \infty$ as G is bounded, we are done.

Write $w \sim w^*$ whenever $w - w^* \in H_0^1(G)$. Then, for all $[w] \in D^1\left(\mathbb{R}^N\right)/H_0^1(G)$, $u := w|_{\overline{U}} \in E^1(U)$ is uniquely defined. Yet, for all $u \in E^1(U)$, (3.11) derives some interior extension, say, $\hat{u}$, in $H^1(G)$ such that $\gamma u = \gamma \hat{u}$ on ∂U, so that, via (3.2), $w := u + \hat{u} \in D^1\left(\mathbb{R}^N\right)$ and $[w]$ is uniquely determined by u. Note here, the notation $[\,\cdot\,]$ denotes the equivalence classes in the quotient group.

Finally, we shall make $E^1(U)$ a Hilbert space. Remember that, for any bounded region Ω with $\partial\Omega$ compact and Lipschitz, $H^1(\Omega)$ is again a real Hilbert function space with respect to the ∂-inner product (see [2, corollary 6.2])

$$\langle u, v\rangle_{\partial,\Omega} := \int_\Omega \nabla u \cdot \nabla v \, dx + \int_{\partial\Omega} \gamma u \cdot \gamma v \, d\sigma, \tag{3.16}$$

and the associated norm is denoted $||u||_{\partial,\Omega}$. Besides, the two norms $||u||_{H^1(\Omega)}$ and

$||u||_{\partial,\Omega}$ are in fact equivalent on $H^1(\Omega)$. Inspired by this, and being aware of (3.11) implying $\gamma(E^1(U)) \subseteq L^2(\partial U, d\sigma)$, we define the ∂-inner product

$$\langle u, v\rangle_{\partial,U} := \int_U \nabla u \cdot \nabla v \, dx + \int_{\partial U} \gamma u \cdot \gamma v \, d\sigma, \tag{3.17}$$

along with the associated norm $||u||_{\partial,U}$, on $E^1(U)$. Then, one has

Theorem 3.3. *Under our general assumption **(B1)**, $E^1(U)$ is a real Hilbert function space with respect to the ∂-inner product (3.17).*

Proof. Obviously, $E^1(U)$ is a vector space as shown in (3.15). Besides, as $\nabla u \equiv 0$ and condition **(A1)** yields $u \equiv 0$, $||\cdot||_{\partial,U}$ is indeed a norm on $E^1(U)$.

Let $\{u_k\}_{k=1}^{\infty}$ be a Cauchy sequence in $E^1(U)$. That is, for each $\varepsilon > 0$, there exists a positive integer $K \in \mathbb{N}$ such that, for all $k_1, k_2 > K$,

$$||u_{k_1} - u_{k_2}||_{\partial,U} \leq \varepsilon, \tag{3.18}$$

from which it follows simultaneously that

$$||\nabla u_{k_1} - \nabla u_{k_2}||_{L^2(U;\mathbb{R}^N)} \leq \varepsilon \text{ and } ||\gamma u_{k_1} - \gamma u_{k_2}||_{2,\partial U} \leq \varepsilon. \tag{3.19}$$

For all $u \in E^1(U)$, define $\tilde{u}$ to be the unique weak solution for

$$\Delta\tilde{u} = 0 \text{ in } G, \text{ subject to } \gamma\tilde{u} = \gamma u \text{ on } \partial U. \tag{3.20}$$

The condition $\gamma\tilde{u} = \gamma u \in H^{\frac{1}{2}}(\partial U, d\sigma)$, via (3.11) and [5, identity (3.2) and theorem 6.2], ensures the existence of our solution in $H^1(G)$. Call $\tilde{u}$ the *interior harmonic* *extension* of u over G, and write $w := \begin{cases} \tilde{u} & \text{on } G, \\ u & \text{on } U. \end{cases}$ Thereby, $\nabla w = \begin{cases} \nabla\tilde{u} & \text{on } G \\ \nabla u & \text{on } U \end{cases}$ is in $L^2\left(\mathbb{R}^N;\mathbb{R}^N\right)$. As G is bounded, w will be in $D^1\left(\mathbb{R}^N\right)$ by (3.2).

Now, let $\tilde{u}_k$ be the interior harmonic extension of u_k over G, and accordingly set $w_k := \begin{cases} \tilde{u}_k & \text{on } G, \\ u_k & \text{on } U, \end{cases}$ for every $k = 1, 2, \ldots$. Applying *Sobolev imbedding theorem for gradients* (see Lieb and Loss [24, theorem 8.3]) then leads to, for some constant $C_N := \pi \sqrt[N]{4\pi}\,\frac{N(N-2)}{4}\left(\Gamma\left(\frac{N+1}{2}\right)\right)^{-\frac{2}{N}}$ relying only on N,

$$\begin{aligned} &||w_{k_1} - w_{k_2}||^2_{2_S,\mathbb{R}^N} \leq C_N\,||\nabla w_{k_1} - \nabla w_{k_2}||^2_{L^2(\mathbb{R}^N;\mathbb{R}^N)} \\ =& C_N\left(\int_U \left|\nabla u_{k_1} - \nabla u_{k_2}\right|^2 dx + \int_G \left|\nabla\tilde{u}_{k_1} - \nabla\tilde{u}_{k_2}\right|^2 dx\right) \\ =& C_N\left(\int_U \left|\nabla u_{k_1} - \nabla u_{k_2}\right|^2 dx + \int_{\partial U} \left(\gamma u_{k_1} - \gamma u_{k_2}\right)\left(D_\nu\tilde{u}_{k_1} - D_\nu\tilde{u}_{k_2}\right) d\sigma\right), \end{aligned} \tag{3.21}$$

where (2.4) was used to derive the last identity above. This shows that $\{w_k\}_{k=1}^\infty$ and $\{\nabla w_k\}_{k=1}^\infty$ are Cauchy sequences in $L^{2_S}\left(\mathbb{R}^N\right)$ and $L^2\left(\mathbb{R}^N;\mathbb{R}^N\right)$, respectively, by (3.19), as $\tilde{u}_k \in H^1(G)$ implies that $D_\nu\tilde{u}_k$ is in $H^{-\frac{1}{2}}(\partial U, d\sigma)$, the dual space of $H^{\frac{1}{2}}(\partial U, d\sigma)$ with respect to the boundary L^2-inner product, for each $k = 1, 2, \ldots$ (see appendix, and also see Auchmuty [4, section 6] and Brezis [15, pp136-137]). Here, D_ν is the unit, outward, normal differential operator.

The completeness of the spaces $L^{2_S}\left(\mathbb{R}^N\right)$ and $L^2\left(\mathbb{R}^N;\mathbb{R}^N\right)$ thereby yields the existence of a function $\mathtt{w} \in L^{2_S}\left(\mathbb{R}^N\right)$ and a vector function $\vec{\mathbf{v}} := \left(\mathtt{v}^1, \mathtt{v}^2, \ldots, \mathtt{v}^N\right) \in L^2\left(\mathbb{R}^N;\mathbb{R}^N\right)$, such that $||w_k - \mathtt{w}||_{2_S,\mathbb{R}^N} \to 0$ and $||\nabla w_k - \vec{\mathbf{v}}||_{L^2(\mathbb{R}^N;\mathbb{R}^N)} \to 0$ when $k \to \infty$, respectively. As a consequence, for all $\varphi \in C_c^1\left(\mathbb{R}^N\right)$, standard argument about weak derivatives derives that

$$\begin{aligned}\int_{\mathbb{R}^N} \mathtt{w}\,(D_j\varphi)\,dx &= \lim_{k\to\infty}\int_{\mathbb{R}^N} w_k\,(D_j\varphi)\,dx \\ = -\lim_{k\to\infty}\int_{\mathbb{R}^N} (D_j w_k)\,\varphi\,dx &= -\int_{\mathbb{R}^N} \mathtt{v}^j\varphi\,dx,\end{aligned} \tag{3.22}$$

which implies that $\nabla \mathtt{w} = \vec{\mathbf{v}} \in L^2\left(\mathbb{R}^N;\mathbb{R}^N\right)$ in the weak sense, where D_j is the j-th weak partial differential operator for each $j = 1, 2, \ldots, N$.

Now, $\mathtt{w} \in L^{2_S}\left(\mathbb{R}^N\right)$ shows $\mathtt{w} \in L^1_{loc}\left(\mathbb{R}^N\right)$, and, from (3.3), $\mathtt{w}$ vanishes at infinity in the sense of **(A1)**. As a result, $\mathtt{w}$ is in $D^1\left(\mathbb{R}^N\right)$. So, one has $u := \mathtt{w}|_{\overline{U}} \in E^1(U)$ from (3.14). Applying the *compact trace theorem* on G (see [21, theorem 1.5.1.10]), together with *Hölder*'s *inequality*, we have that, for this function $\tilde{u} := \mathtt{w}|_{\overline{G}}$, and for other two vector functions $\nabla u := \vec{\mathbf{v}}|_{\overline{U}}$ and $\nabla\tilde{u} := \vec{\mathbf{v}}|_{\overline{G}}$,

$$\begin{aligned}&||u_k - u||^2_{\partial,U} = ||\nabla u_k - \nabla u||^2_{L^2(U;\mathbb{R}^N)} + ||\gamma u_k - \gamma u||^2_{2,\partial U} \\ &\leq ||\nabla u_k - \nabla u||^2_{L^2(U;\mathbb{R}^N)} + C_G\left(||\tilde{u}_k - \tilde{u}||^2_{2,G} + ||\nabla\tilde{u}_k - \nabla\tilde{u}||^2_{L^2(G;\mathbb{R}^N)}\right) \\ &\leq (1 + C_G)\,||\nabla w_k - \vec{\mathbf{v}}||_{L^2(\mathbb{R}^N;\mathbb{R}^N)} + C_G\left(\mathcal{M}(G)\right)^{\frac{2}{N}}||\tilde{u}_k - \tilde{u}||^2_{2_S,G} \\ &\leq (1 + C_G)\,||\nabla w_k - \vec{\mathbf{v}}||_{L^2(\mathbb{R}^N;\mathbb{R}^N)} + C_G\left(\mathcal{M}(G)\right)^{\frac{2}{N}}||w_k - \mathtt{w}||^2_{2_S,\mathbb{R}^N} \to 0\end{aligned} \tag{3.23}$$

as $k \to \infty$, where $C_G > 0$ is a constant depending only on G.

Thus, $E^1(U)$ is complete with respect to the norm $||\cdot||_{\partial,U}$. $\square$

Remark 3.4. *In general, though one has $||\cdot||_{\partial,U} \leq C\,||\cdot||_{H^1(U)}$ for some constant $C > 0$ that depends only on U (see [10, theorem 3.1]), $H^1(U)$ is not complete with respect to $||\cdot||_{\partial,U}$. This can be shown via the function $f(x) = |x|^{2-N}$, using the cut-off functions $\chi_n(x)$ by (4.12) and the sequence $\{u_{n+1} := \chi_n f\}_{n=1}^{\infty}$.*

Remark 3.5. *Adapting the above arguments, $D^1\left(\mathbb{R}^N\right)$ can be shown a real Hilbert function space with respect to $\langle\cdot,\cdot\rangle_\nabla$, the gradient $L^2\left(\mathbb{R}^N;\mathbb{R}^N\right)$-inner product,*

$$\langle w, w^*\rangle_\nabla := \int_{\mathbb{R}^N} \nabla w \cdot \nabla w^* \, dx, \tag{3.24}$$

with its associated norm $||w||_\nabla$. See also Auchmuty [7, sections 2 and 3].

Remark 3.6. *From the above discussions, along with the decomposition of $H^1(G)$, as described by Auchmuty (see [5, identity (3.2) and theorem 6.2]), that is,*

$$H^1(G) = H_0^1(G) \oplus_{\partial,G} \mathscr{H}(G), \tag{3.25}$$

with $\mathscr{H}(G)$ being the subspace of $H^1(G)$ of functions harmonic on G, we have

$$D^1\left(\mathbb{R}^N\right) = E^1(U) \oplus_\nabla H^1(G) = E^1(U) \oplus_\nabla \left[H_0^1(G) \oplus_{\partial,G} \mathscr{H}(G)\right]. \tag{3.26}$$

Here and henceforth, we use different subscripts under the notation $\oplus$ to indicate the direct sums are related to the corresponding inner products.

Chapter 4

Approximation of Function Spaces

4.1 The Approximation of $H^1(U)$

Below, we approximate the exterior region U and the space $H^1(U)$ accordingly as an illustration to that of $E^1(U)$, which will be given in the next section.

Fix a $r > \max\{1, r_0\}$, so that $\mathbb{R}^N \setminus U \Subset B_r$. For every $n = 1, 2, \ldots$, define the bounded region $U_n := U \cap B_{r^n}$, having a compact, Lipschitz boundary $\partial U_n = \partial U \,\dot\cup\, S_{r^n}$, and correspondingly define the *truncated finite energy space*

$$E^1(U_n) := \left\{u \in H^1(U) : u \equiv 0 \text{ on } U \setminus U_n\right\}. \tag{4.1}$$

In consequence, we have, via the very definition,

$$E^1(U_1) \subseteq \cdots \subseteq E^1(U_n) \subseteq E^1(U_{n+1}) \subseteq \cdots \subseteq H^1(U). \tag{4.2}$$

Here and henceforth, by abuse of notations, we write U_n instead of U_{r^n}.

Denote the restriction of $E^1(U_n)$ on U_n by $H^1_{\hat{0}}(U_n)$.

Obviously, $H^1_{\hat{0}}(U_n)$ is a subspace of $H^1(U_n)$ of functions whose traces are zero on S_{r^n}. Take $u \in H^1(U_n)$ to be such that $\gamma u \equiv 0$ on S_{r^n}. Via zero extension to infinity, it is then in $E^1(U_n)$. So, u itself is in $H^1_{\hat{0}}(U_n)$. $H^1_{\hat{0}}(U_n)$ is also closed in $H^1(U_n)$. Actually, let $\{u_k\}_{k=1}^{\infty}$ be a Cauchy sequence in $H^1_{\hat{0}}(U_n)$ with respect to $||\cdot||_{H^1(U_n)}$. Recall $||\cdot||_{H^1(U_n)}$ and $||\cdot||_{\partial,U_n}$ are equivalent on $H^1(U_n)$. The completeness of $H^1(U_n)$ yields a function $u \in H^1(U_n)$ such that

$$\int_{U_n} |\nabla u_k - \nabla u|^2 \, dx + \int_{\partial U} (\gamma u_k - \gamma u)^2 \, d\sigma + \int_{S_{r^n}} (\gamma u)^2 \, d\sigma \to 0 \tag{4.3}$$

when $k \to \infty$. Therefore, $\gamma u \equiv 0$ on S_{r^n}, so that u is in $H^1_{\hat{0}}(U_n)$.

Consequently, $H^1_{\hat{0}}(U_n)$ is the maximal subspace of $H^1(U_n)$ of all functions that have zero traces on S_{r^n}. So, $H^1_0(U_n) \subsetneq H^1_{\hat{0}}(U_n)$ follows accordingly, where $H^1_0(U_n)$ denotes the closure of the set $C^1_c(U_n)$ with respect to $||\cdot||_{H^1(U_n)}$.

Our next step is to prove the following approximation result.

Proposition 4.1. *Given $u \in H^1(U)$, there exists a sequence $\{u_n\}_{n=1}^{\infty}$ of functions, with $u_n \in E^1(U_n)$ for each $n \geq 1$, such that $\lim_{n\to\infty} ||u_n - u||_{H^1(U)} = 0$.*

Proof. Define, for every $n = 1, 2, \ldots$, and for all $x \in \mathbb{R}^N$,

$$f_n(|x|) := \begin{cases} 0 & \text{on } B_{r^n} \cup \left\{\mathbb{R}^N \setminus \overline{B}_{r^{n+1}}\right\}, \\ \\ \exp\left(\frac{1}{(|x|-r^n)(|x|-r^{n+1})}\right) & \text{on } \overline{B}_{r^{n+1}} \setminus B_{r^n}, \end{cases} \tag{4.4}$$

and correspondingly write $X_n(|x|) := \frac{\int_{|x|}^{+\infty} f_n(t)\,dt}{\int_{-\infty}^{+\infty} f_n(t)\,dt}$.

Let $\mathcal{X}_n(x)$ be the restriction of $X_n(|x|)$ on $\overline{U}$. We then easily see that $\mathcal{X}_n$ is a decreasing, smooth function, lying in between $[0,1]$, such that

$$\begin{cases} \mathcal{X}_n \equiv 1 & \text{on } \overline{U}_n, \\ \\ \mathcal{X} \equiv 0 & \text{on } U \setminus U_{n+1}, \end{cases} \tag{4.5}$$

and such that $||\nabla \mathcal{X}_n||_{\infty,U}$ is bounded, independent of n.

Now, for all $u \in H^1(U)$, write $u_{n+1} := \mathcal{X}_n u$. Then, u_{n+1} is in $E^1(U_{n+1})$ since $u_{n+1} \equiv 0$ on $U \setminus U_{n+1}$. Besides, $||u_{n+1} - u||_{2,U} \leq 2\,||u||_{2,U\setminus U_n} \to 0$ as $n \to \infty$, for $u \in L^2(U)$. Moreover, from *Cauchy*'s *inequality*, we have

$$\begin{aligned} &\int_U |\nabla u_{n+1} - \nabla u|^2\,dx = \int_U |u\nabla\mathcal{X}_n + \mathcal{X}_n\nabla u - \nabla u|^2\,dx \\ &= \int_{U\setminus U_{n+1}} |\nabla u|^2\,dx + \int_{\overline{U}_{n+1}\setminus U_n} |u\nabla\mathcal{X}_n + (\mathcal{X}_n - 1)\nabla u|^2\,dx \\ &\leq \int_{U\setminus U_n} |\nabla u|^2\,dx + \int_{\overline{U}_{n+1}\setminus U_n} |u\nabla\mathcal{X}_n|^2 + 2\int_{\overline{U}_{n+1}\setminus U_n} |u\nabla\mathcal{X}_n \cdot \nabla u|\,dx \\ &\leq 2\left(\int_{U\setminus U_n} |\nabla u|^2\,dx + ||\nabla\mathcal{X}_n||_{\infty,U}^2 \int_{U\setminus U_n} u^2\,dx\right) \to 0 \end{aligned} \tag{4.6}$$

as $n \to \infty$, for $u \in H^1(U)$. Thus, $||u_{n+1} - u||_{H^1(U)} \to 0$ as $n \to \infty$. □

Let $C_\omega^1\left(\overline{U}\right)$, as before, be the restriction of $C_c^1\left(\mathbb{R}^N\right)$ on $\overline{U}$. That is,

$$C_\omega^1\left(\overline{U}\right) := \left\{\psi : \psi = \varphi|_{\overline{U}} \text{ for some } \varphi \in C_c^1\left(\mathbb{R}^N\right)\right\}. \tag{4.7}$$

By remarks 2.1 and 2.2, $C_\omega^1\left(\overline{U}\right)$ is dense in $H^1(U)$ with respect to $||\cdot||_{H^1(U)}$.

4.2 The Approximation of $E^1(U)$

In this section, we approximate the finite energy space $E^1(U)$.

First, observe that the truncated finite energy space $E^1(U_n)$ equals

$$E^1(U_n) = \left\{u \in E^1(U) : u \equiv 0 \text{ on } U \setminus U_n\right\} \tag{4.8}$$

via condition **(B2)** and (3.2), when restricted on U_n. So, we also have

$$E^1(U_1) \subseteq \cdots \subseteq E^1(U_n) \subseteq E^1(U_{n+1}) \subseteq \cdots \subseteq E^1(U). \tag{4.9}$$

Similarly, we can derive the approximation result below.

Proposition 4.2. *Given $u \in E^1(U)$, there exists a sequence $\{u_n\}_{n=1}^{\infty}$ of functions, with $u_n \in E^1(U_n)$ for each $n \geq 1$, such that $\lim\limits_{n\to\infty} ||u_n - u||_{\partial,U} = 0$.*

Proof. Define, for every $f_n(|x|)$, given by (4.4), with $n = 1, 2, \ldots,$

$$F_n(|x|) := \begin{cases} \frac{\int_{|x|}^{+\infty} f_n(t)dt}{\int_{-\infty}^{+\infty} f_n(t)dt} & \text{on } B_{r^n+\epsilon_n} \cup \left\{\mathbb{R}^N \setminus \overline{B}_{r^{n+1}-\epsilon_n}\right\}, \\ \\ \frac{\int_{r^n+\epsilon_n}^{+\infty} f_n(t)dt}{\int_{-\infty}^{+\infty} f_n(t)dt} - \frac{(|x|-r^n-\epsilon_n)\int_{r^n+\epsilon_n}^{r^{n+1}-\epsilon_n} f_n(t)dt}{(r^{n+1}-r^n-2\epsilon_n)\int_{-\infty}^{+\infty} f_n(t)dt} & \text{on } \overline{B}_{r^{n+1}-\epsilon_n} \setminus B_{r^n+\epsilon_n}, \end{cases} \tag{4.10}$$

where $\epsilon_n = O\left(\frac{1}{r^{n+1}\log(r^{n+1})}\right) > 0$ is a constant so chosen that

$$\exp\left(\frac{1}{\epsilon_n\left(r^n + \epsilon_n - r^{n+1}\right)}\right) = \frac{\int_{r^n+\epsilon_n}^{r^{n+1}-\epsilon_n} f_n(t)dt}{r^{n+1} - r^n - 2\epsilon_n}. \tag{4.11}$$

Let $\chi_n(x)$ be the restriction of $F_n(|x|)$ on $\overline{U}$. One then easily sees that χ_n is a decreasing, smooth function, lying in between $[0,1]$, such that

$$\begin{cases} \chi_n \equiv 1 & \text{on } \overline{U}_n, \\ \\ \chi_n \equiv 0 & \text{on } U \setminus U_{n+1}, \\ \\ \|\nabla \chi_n\|_{\infty,U} & \text{is exactly } O\left(\frac{1}{r^{n+1}}\right). \end{cases} \tag{4.12}$$

For all $u \in E^1(U)$, set $u_{n+1} := \chi_n u$. So, u_{n+1} is in $E^1(U_{n+1})$ via the boundedness of $\nabla\chi_n$ and (3.2), as $u_{n+1} \equiv 0$ on $U \setminus U_{n+1}$. In addition, $\gamma u_{n+1} = \gamma u$ on ∂U. From proposition 3.1, and *Cauchy*'s and *Hölder*'s *inequalities*, we have

$$\begin{aligned}
&\int_U |\nabla u_{n+1} - \nabla u|^2\, dx = \int_U |u\nabla\chi_n + \chi_n \nabla u - \nabla u|^2\, dx \\
&= \int_{U\setminus\overline{U}_{n+1}} |\nabla u|^2\, dx + \int_{U_{n+1}\setminus\overline{U}_n} |u\nabla\chi_n + (\chi_n - 1)\nabla u|^2\, dx \\
&\le \int_{U\setminus\overline{U}_{n+1}} |\nabla u|^2\, dx + 2\int_{U_{n+1}\setminus\overline{U}_n} \left(|u\nabla\chi_n|^2 + |\nabla u|^2\right) dx \\
&\le 2\int_{U\setminus\overline{U}_n} |\nabla u|^2\, dx + 2\,\|\nabla\chi_n\|^2_{\infty,U} \int_{U_{n+1}\setminus\overline{U}_n} |u|^2\, dx \\
&\le 2\int_{U\setminus\overline{U}_n} |\nabla u|^2\, dx + 2\,\|\nabla\chi_n\|^2_{\infty,U} \left(\int_{U_{n+1}\setminus\overline{U}_n} dx\right)^{\frac{2}{N}} \left(\int_{U_{n+1}\setminus\overline{U}_n} |u|^{2_S}\, dx\right)^{\frac{N-2}{N}} \\
&\le 2\,\|\nabla u\|^2_{L^2\left(U\setminus\overline{U}_n;\mathbb{R}^N\right)} + 2\,\|\nabla\chi_n\|^2_{\infty,U} \left(\mathcal{M}\left(U_{n+1}\setminus\overline{U}_n\right)\right)^{\frac{2}{N}} \|u\|^2_{2_S,U\setminus\overline{U}_n} \to 0
\end{aligned} \tag{4.13}$$

when $n \to \infty$, as $u \in E^1(U)$ implies that $\|u\|^2_{2_S,U\setminus\overline{U}_n} \to 0$ and $\|\nabla u\|^2_{L^2\left(U\setminus\overline{U}_n;\mathbb{R}^N\right)} \to 0$, respectively, when $n \to \infty$, while, on the other hand, as $\|\nabla\chi_n\|_{\infty,U} = O\left(\frac{1}{r^{n+1}}\right)$ yields that $\|\nabla\chi_n\|_{\infty,U}\left(\mathcal{M}\left(U_{n+1}\setminus\overline{U}_n\right)\right)^{\frac{1}{N}} = O(1)$, independent of r, when $n \to \infty$.

In consequence, it follows that $||u_{n+1}-u||_{\partial,U} \to 0$ when $n \to \infty$. □

Propositions 4.1 and 4.2 tell us that $E^1(U_n)$ exhausts $H^1(U)$ with respect to $||\cdot||_{H^1(U)}$, and $E^1(U)$ instead with respect to $||\cdot||_{\partial,U}$, as $n \to \infty$.

Recall that $D^1(\mathbb{R}^N)$ is the finite energy space on $\mathbb{R}^N$. One has

Corollary 4.3. *Take $\{V_j\}_{j=1}^{\infty}$ to be any class of bounded regions satisfying $V_j \Subset V_{j+1}$ and $\mathbb{R}^N = \bigcup_{j=1}^{\infty} V_j$, such that ∂V_j is compact and Lipschitz for every $j = 1, 2, \ldots$. Then, for all $w \in D^1(\mathbb{R}^N)$, there exists a sequence $\{w_j\}_{j=1}^{\infty}$ of functions, with $w_j \in H_0^1(V_j)$ for each $j \geq 1$, such that $\lim_{j\to\infty} ||w_j - w||_{\nabla} = 0$. Here, $||\cdot||_{\nabla}$ (see remark 3.5) denotes the gradient $L^2(\mathbb{R}^N;\mathbb{R}^N)$-norm.*

Proof. As discussed ahead of (3.13), via zero extension over $\mathbb{R}^N$, $H_0^1(V_j)$ becomes a subspace of $D^1(\mathbb{R}^N)$ with respect to the norm $||\cdot||_{\nabla}$. For all j sufficiently large, there are two integers $n_{j_2} > n_{j_1}$, with $B_{r^{n_{j_1}}} \Subset V_j \Subset B_{r^{n_{j_2}}}$, such that

$$H_0^1(B_{r^{n_{j_1}}}) \subseteq H_0^1(V_j) \subseteq H_0^1(B_{r^{n_{j_2}}}). \tag{4.14}$$

Now, a slightly modified proof of proposition 4.2 shows that the space $D^1(\mathbb{R}^N)$ can be approximated through the spaces $H_0^1(B_{r^n})$ as $n \to \infty$ with respect to $||\cdot||_{\nabla}$. Actually, for all $w \in D^1(\mathbb{R}^N)$, write $w_{n+1} := F_n w \in H_0^1(B_{r^{n+1}})$. Proposition 3.1, the fact $||\nabla F_n||_{\infty,U} = O\left(\frac{1}{r^{n+1}}\right)$ and the estimate (4.13) then yield it.

Therefore, along with (4.14), this gives us the desired limit. □

Remark 4.4. *What this result really says is, with respect to $||\cdot||_{\nabla}$, $C_c^1(\mathbb{R}^N)$ is dense in $D^1(\mathbb{R}^N)$, as $D^1(\mathbb{R}^N)$ can be approximated by $H_0^1(B_{r^n})$ and $C_c^1(B_{r^n})$ is dense in $H_0^1(B_{r^n})$, in view of Poincaré's inequality.*

Recall that $C_\omega^1\left(\overline{U}\right)$ denotes the restriction of $C_c^1\left(\mathbb{R}^N\right)$ on $\overline{U}$. $C_\omega^1\left(\overline{U}\right)$ is dense in $E^1(U)$ with respect to $||\cdot||_{\partial,U}$, which follows from remarks 2.1 and 2.2 again, together with an estimate like (3.23). We shall show it below.

Let u be a function in $E^1(U)$. For each $\varepsilon > 0$, there exists a $u_n \in E^1(U_n)$, for sufficiently large n, such that $||u_n - u||_{\partial,U} \leq \frac{\varepsilon}{2}$. Restricting u_n on U_n and using the same notation leads to $u_n \in H_{\widetilde{0}}^1(U_n)$. Remarks 2.1 and 2.2 (also see (3.11)) then give a $w_n \in H_0^1(B_{r^n})$, which is an extension of u_n over B_{r^n}, as $\gamma u_n \equiv 0$ on S_{r^n} and $U_n \subseteq B_{r^n}$. Therefore, there exists a sequence $\{\varphi_k\}_{k=1}^\infty$ of functions in $C_c^1(B_{r^n})$ such that $||\nabla\varphi_k - \nabla w_n||_\nabla \to 0$ as $k \to \infty$, by *Poincaré*'s *inequality* and zero extension outside B_{r^n} to $\mathbb{R}^N$. Let ψ_k be the restriction of φ_k on $\overline{U}$. Then,

$$\begin{aligned}
&||\psi_k - u_n||_{\partial,U}^2 \leq ||\nabla\varphi_k - \nabla w_n||_{L^2(U;\mathbb{R}^N)}^2 + ||\gamma\varphi_k - \gamma w_n||_{2,\partial U}^2 \\
\leq &||\nabla\varphi_k - \nabla w_n||_{L^2(U;\mathbb{R}^N)}^2 + C_G\left(||\varphi_k - w_n||_{2,G}^2 + ||\nabla\varphi_k - \nabla w_n||_{L^2(G;\mathbb{R}^N)}^2\right) \\
\leq &\max\{1, C_G\}\,||\nabla\varphi_k - \nabla w_n||_{L^2(B_{r^n};\mathbb{R}^N)}^2 + C_G\,||\varphi_k - w_n||_{2,B_{r^n}}^2 \\
\leq &C_{B_{r^n}}\,||\nabla\varphi_k - \nabla w_n||_{L^2(B_{r^n};\mathbb{R}^N)}^2 = C_{B_{r^n}}\,||\nabla\varphi_k - \nabla w_n||_\nabla^2 \to 0
\end{aligned} \tag{4.15}$$

as $k \to \infty$, where the *compact trace theorem* on G and *Poincaré*'s *inequality* were applied for deriving this estimate, $C_G > 0$ is the constant given below (3.23), and $C_{B_{r^n}} > 0$ is a constant relying only upon B_{r^n}. As a consequence, $||\psi_k - u||_{\partial,U} \leq \varepsilon$ follows immediately when k is sufficiently large.

4.3 The *Gauss-Green Theorem*

As an application, we have the following version *Gauss-Green theorem*.

Theorem 4.5. *Let $u \in E^1(U)$ be such that its Laplacian Δu exists and is in the intersection space $L^2_{loc}\left(\overline{U}\right) \cap L^{\frac{2N}{N+2}}(U)$. Then, for all $v \in E^1(U)$, one has*

$$\int_U [(\Delta u)\, v + \nabla u \cdot \nabla v]\, dx = \int_{\partial U} (D_\nu u)\, \gamma v\, d\sigma. \tag{4.16}$$

In particular, when u is a function in $H^1(U)$ such that its Laplacian Δu exists and is in the space $L^2(U)$, this identity still holds for all $v \in H^1(U)$.

Proof. On each bounded region U_n, in view of our assumptions, we have

$$\int_{U_n} [(\Delta u)\, \psi + \nabla u \cdot \nabla \psi]\, dx = \int_{\partial U} (D_\nu u)\, \gamma \psi\, d\sigma + \int_{S_{r^n}} (D_\nu u)\, \gamma \psi\, d\sigma \tag{4.17}$$

by (2.4) for such a $u \in E^1(U)$ described as above, so that, letting $n \to \infty$ leads to

$$\int_U [(\Delta u)\, \psi + \nabla u \cdot \nabla \psi]\, dx = \int_{\partial U} (D_\nu u)\, \gamma \psi\, d\sigma \tag{4.18}$$

for all $\psi \in C^1_\omega\left(\overline{U}\right)$. Being aware of the fact $u \in L^{2s}(U)$ and the density of $C^1_\omega\left(\overline{U}\right)$ in $E^1(U)$ with respect to $||\cdot||_{\partial,U}$, our desired result then follows.

On the other hand, for such a $u \in H^1(U)$ as described in our hypothesis, the identity (4.18) still holds for all $\psi \in C^1_\omega\left(\overline{U}\right)$. Noticing now the density of $C^1_\omega\left(\overline{U}\right)$ in $H^1(U)$ with respect to $||\cdot||_{H^1(U)}$, we finally finishes the proof. □

Corollary 4.6. *The respective weak solutions of the systems below,*

$$\begin{cases} -\Delta u = 0 \text{ in } U, \quad \text{subject to } \gamma u = \eta_1 \text{ on } \partial U, \\ \text{or} \\ -\Delta u = 0 \text{ in } U, \quad \text{subject to } D_\nu u + b(\gamma u) = \eta_2 \text{ on } \partial U, \end{cases} \tag{4.19}$$

in our space $E^1(U)$, if exist, are unique, where $b \geq 0$ is a constant.

In addition, the respective weak solutions of the systems below,

$$\begin{cases} \mu^2 u - \Delta u = 0 \textit{ in } U, \quad \textit{subject to } \gamma u = \eta_3 \textit{ on } \partial U, \\ \textit{or} \\ \mu^2 u - \Delta u = 0 \textit{ in } U, \quad \textit{subject to } D_\nu u + b(\gamma u) = \eta_4 \textit{ on } \partial U, \end{cases} \tag{4.20}$$

with some constant $\mu > 0$, in the space $H^1(U)$, if exist, are again unique.

Proof. Take $v = u \in E^1(U)$ to be harmonic in (4.16) such that either $\gamma u \equiv 0$ or $D_\nu u + b(\gamma u) \equiv 0$ on ∂U. We thereby have $\int_U |\nabla u|^2 \, dx = 0$, so that $u \equiv 0$ from condition **(A1)**. This consequently confirms the first statement.

When u is a weak solution of $\mu^2 u - \Delta u = 0$, applying (4.16) yields

$$\int_U \left(\mu^2 uv + \nabla u \cdot \nabla v\right) dx = \int_{\partial U} (D_\nu u)\, \gamma v \, d\sigma, \quad \forall\, v \in H^1(U). \tag{4.21}$$

Suppose again that either $\gamma u \equiv 0$ or $D_\nu u + b(\gamma u) \equiv 0$ on ∂U. Substituting $v = u$ into (4.21) leads to $u \equiv 0$ as $\mu > 0$. This finally finishes our proof. $\square$

Chapter 5

Steklov Eigenproblems in the Function Space $H^1_{\widehat{0}}(U_n)$

5.1 The Regularized Harmonic Case

In this section, we study in detail the regularized harmonic Steklov eigenproblems in the truncated finite energy space $H^1_{\widehat{0}}(U_n)$, using standard variational principles and convex analysis, whereas simply outline the parallel results in the next section for the harmonic Steklov eigenproblems, again in $H^1_{\widehat{0}}(U_n)$.

In the following, we shall find the weak solutions of the following *mixed Dirichlet-Robin type* eigenvalue problem in $H^1(U_n)$

$$\begin{cases} \mathcal{L}_1(u) := u - \Delta u = 0 \text{ in } U_n, \text{ subject to} \\ \\ D_\nu u = \tau u \text{ on } \partial U, \text{ and } \gamma u = 0 \text{ on } S_{r^n}. \end{cases} \tag{5.1}$$

On the bounded region U_n having a nice boundary $\partial U_n = \partial U \,\dot{\cup}\, S_{r^n}$, the *Gauss-Green theorem* holds. In consequence, we are going to find the non-trivial solutions in $H^1_{\hat{0}}(U_n) \times (0,\infty)$ that satisfies the following identity

$$\int_{U_n} (\nabla u \cdot \nabla v + uv)\, dx - \tau \int_{\partial U} \gamma u \cdot \gamma v \, d\sigma = 0, \quad \forall\, v \in H^1_{\hat{0}}(U_n)\,. \tag{5.2}$$

We call our problem *regularized harmonic* due to the operator $\mathcal{L}_1(u) = u - \Delta u$, yet, the results are exactly the same if we instead consider the operator $\mathcal{L}_\mu(u) := \mu^2 u - \Delta u$ for $\mu > 0$ and using the weighted H^1_μ-inner product. To avoid blurring the essence, we simply consider the standard case where $\mu = 1$.

Let $\mathbf{K}$ be the closed unit ball in $H^1_{\hat{0}}(U_n)$ with respect to the standard H^1-norm, given by (2.2). That is, $\mathbf{K} \subseteq H^1_{\hat{0}}(U_n)$ is such that

$$\mathbf{K} := \left\{ u \in H^1_{\hat{0}}(U_n) : \int_{U_n} \left(u^2 + |\nabla u|^2\right) dx \le 1 \right\}. \tag{5.3}$$

Define the functional $\mathcal{T} : H^1(U) \to [0,\infty)$ by

$$\mathcal{T}(u) := \int_{\partial U} (\gamma u)^2 \, d\sigma. \tag{5.4}$$

Note here, $\mathcal{T}$ is also well-defined by (4.2) for u in $H^1_{\hat{0}}(U_n)$. Consider the variational principle $\left(\mathfrak{R}\mathfrak{h}\mathfrak{S}_{1,n}\right)$ of maximizing $\mathcal{T}$ on $\mathbf{K}$, and write

$$\kappa_{1,n} := \sup_{u \in \mathbf{K}} \mathcal{T}(u). \tag{5.5}$$

$\mathbf{K}$ is a bounded, closed, convex subset of $H^1_{\hat{0}}(U_n)$, which enables us to prove the following existence result for solutions of $\left(\mathfrak{R}\mathfrak{h}\mathfrak{S}_{1,n}\right)$.

Theorem 5.1. *There exists some maximizer* $\mathfrak{u}_{1,n}$ *(so does* $-\mathfrak{u}_{1,n}$*) of* $\mathcal{T}$ *in* $\mathbf{K}$ *with* $||\mathfrak{u}_{1,n}||_{H^1(U_n)} = 1$*, which is an eigenfunction for the problem (5.2) corresponding to the least positive eigenvalue* $\tau_{1,n}$ *with* $\tau_{1,n} := \frac{1}{\kappa_{1,n}}$.

Proof. As $\mathbf{K}$ is bounded, closed and convex, it is weakly compact in $H^1_{\circ}(U_n)$. Also, the *compact trace theorem* on U_n implies that $\mathcal{T}$ is weakly continuous on the space $H^1(U_n)$, and so is it on $H^1_{\circ}(U_n)$. Hence, $\mathcal{T}$ attains its supremum, technically as a maximum, at at least one function, say, $\mathfrak{u}_{1,n} \in \mathbf{K}$, such that $\kappa_{1,n} = \mathcal{T}(\mathfrak{u}_{1,n}) > 0$ is finite. If $||\mathfrak{u}_{1,n}||_{H^1(U_n)} < 1$, we set $c := ||\mathfrak{u}_{1,n}||^{-1}_{H^1(U_n)} > 1$, so that $||c\mathfrak{u}_{1,n}||_{H^1(U_n)} = 1$; yet, $c\mathfrak{u}_{1,n} \in \mathbf{K}$ and $\mathcal{T}(c\mathfrak{u}_{1,n}) = c^2\mathcal{T}(\mathfrak{u}_{1,n}) > \kappa_{1,n}$. This is a contradiction.

A Lagrangian functional for the above variational principle $\left(\mathfrak{R}\mathfrak{h}\mathfrak{S}_{1,n}\right)$ is given by $\mathcal{F}_1 : H^1_{\circ}(U_n) \times [0,\infty) \to \mathbb{R}$, which is defined as, for some constant $\lambda \geq 0$,

$$\mathcal{F}_1(u,\lambda) := \lambda\left(\int_{U_n}\left(u^2 + |\nabla u|^2\right)dx - 1\right) - \int_{\partial U}(\gamma u)^2\, d\sigma. \tag{5.6}$$

Our problem of maximizing $\mathcal{T}$ on $\mathbf{K}$ is equivalent to finding an inf-sup point of $\mathcal{F}_1$ on its domain. Any such a maximizer is a critical point of $\mathcal{F}_1(\cdot,\lambda)$ on $H^1_{\circ}(U_n)$, that is, $\mathcal{F}_1'(\cdot,\lambda)(v) = 0$ for all $v \in H^1_{\circ}(U_n)$. As a result, we have

$$\lambda\left(\int_{U_n}\left(\nabla\mathfrak{u}_{1,n}\cdot\nabla v + \mathfrak{u}_{1,n}v\right)dx\right) - \int_{\partial U}\gamma\mathfrak{u}_{1,n}\cdot\gamma v\, d\sigma = 0, \quad \forall\, v \in H^1_{\circ}(U_n). \tag{5.7}$$

Letting $v = \mathfrak{u}_{1,n}$ yields $\lambda = \kappa_{1,n}$. Thus, (5.2) holds with $\tau_{1,n} = \frac{1}{\kappa_{1,n}} > 0$.

If $\tau_{1,n}$ is not the least positive eigenvalue of (5.2), there would be a $\ddot{\tau}_{1,n} < \tau_{1,n}$ and some $\ddot{\mathfrak{u}}_{1,n} \in \mathbf{K}$ with $||\ddot{\mathfrak{u}}_{1,n}||_{H^1(U_n)} = 1$ such that (5.2) is satisfied by $(\ddot{\mathfrak{u}}_{1,n}, \ddot{\tau}_{1,n})$.

Yet, this is impossible, for otherwise we would have $\ddot{\kappa}_{1,n} = \frac{1}{\ddot{\tau}_{1,n}} > \frac{1}{\tau_{1,n}} = \kappa_{1,n}$. $\square$

Remark 5.2. *As an immediate application, by homogeneity, it follows that*

$$||\gamma u||_{2,\partial U} \leq \sqrt{\kappa_{1,n}}\, ||u||_{H^1(U_n)}\,, \quad \forall\, u \in H^1_{\circ}\left(U_n\right). \tag{5.8}$$

Given the first k (≥ 1) regularized harmonic Steklov eigenvalues $\{\tau_{1,n}, \ldots, \tau_{k,n}\}$ and an associated set $\{\mathfrak{u}_{1,n}, \ldots, \mathfrak{u}_{k,n}\}$ of $\langle \cdot, \cdot \rangle_{H^1(U_n)}$-orthonormal eigenfunctions, we show below how to find the next pair $(\mathfrak{u}_{k+1,n}, \tau_{k+1,n}) \in H^1_{\circ}\left(U_n\right) \times (0, \infty)$.

For each $k \geq 1$, define

$$\mathbf{K}_k := \Big\{ u \in \mathbf{K} : \langle \gamma u, \gamma \mathfrak{u}_{i,n} \rangle_{2,\partial U} = 0 \ \text{ for } i = 1, 2, \ldots, k \Big\}. \tag{5.9}$$

Noticing $0 < \tau_{1,n} \leq \tau_{2,n} \leq \cdots \leq \tau_{k,n}$, by (5.2), it is equivalent to set

$$\mathbf{K}_k := \Big\{ u \in \mathbf{K} : \langle u, \mathfrak{u}_{i,n} \rangle_{H^1(U_n)} = 0 \ \text{ for } i = 1, 2, \ldots, k \Big\}. \tag{5.10}$$

Consider the variational principle $\left(\mathfrak{R}\mathfrak{h}\mathfrak{S}_{k+1,n}\right)$ of maximizing the functional $\mathcal{T}$ on the subset $\mathbf{K}_k$ of $H^1_{\circ}\left(U_n\right) \subseteq H^1\left(U_n\right)$, and write

$$\kappa_{k+1,n} := \sup_{u \in \mathbf{K}_k} \mathcal{T}(u). \tag{5.11}$$

Theorem 5.3. *There are maximizers $\pm \mathfrak{u}_{k+1,n}$ of $\mathcal{T}$ on $\mathbf{K}_k$ with $||\mathfrak{u}_{k+1,n}||_{\partial, U_n} = 1$. These functions are eigenfunctions of our problem (5.2) associated with the eigenvalue $\tau_{k+1,n}$ such that $\tau_{k+1,n} := \frac{1}{\kappa_{k+1,n}}$. Moreover, $\tau_{k+1,n}$ is the smallest eigenvalue for our problem greater than or equal to $\tau_{k,n}$.*

Proof. Via the *compact trace theorem* on U_n, each of the linear functional $\mathcal{T}_i(u) := \int_{\partial U} \gamma \mathfrak{u}_{i,n} \cdot \gamma u \, d\sigma$ is continuous in $H^1_{\hat{0}}(U_n)$ for $i = 1, 2, \ldots, k$. Thus, $\mathbf{K}_k$ is bounded, closed and convex in $H^1_{\hat{0}}(U_n)$ by (5.9). So, it is weakly compact.

As a result, $\mathcal{T}$ attains its supremum, again as a maximum, at some function, say, $\mathfrak{u}_{k+1,n} \in \mathbf{K}_k$, such that $\kappa_{k+1,n} = \mathcal{T}(\mathfrak{u}_{k+1,n}) > 0$ is finite. Also, $\kappa_{k+1,n} \leq \kappa_{k,n}$ follows, and, similar to our proof of theorem 5.1, $||\mathfrak{u}_{k+1,n}||_{H^1(U_n)} = 1$ holds.

Let $\mathbf{V}_k$ be the linear space spanned by $\{\mathfrak{u}_{1,n}, \mathfrak{u}_{2,n}, \ldots, \mathfrak{u}_{k,n}\}$. A result of Auchmuty (see [3, theorem 2.1]) says $\mathfrak{u}_{k+1,n}$ satisfying the identity below

$$\langle \gamma u, \gamma v \rangle_{2,\partial U} = \langle \lambda u + \omega, v \rangle_{H^1(U_n)}, \quad \forall\, v \in H^1_{\hat{0}}(U_n), \tag{5.12}$$

for a constant $\lambda \geq 0$ and some function ω in $\mathbf{V}_k$. That is,

$$\begin{aligned} &\int_{\partial U} \gamma \mathfrak{u}_{k+1,n} \cdot \gamma v \, d\sigma - \int_{U_n} (\nabla \omega \cdot \nabla v + \omega v)\, dx \\ &= \lambda \int_{U_n} (\nabla \mathfrak{u}_{k+1,n} \cdot \nabla v + \mathfrak{u}_{k+1,n} v)\, dx, \quad \forall\, v \in H^1_{\hat{0}}(U_n). \end{aligned} \tag{5.13}$$

Keep in mind (5.9) and (5.10). Letting $v = \mathfrak{u}_{k+1,n}$ yields $\lambda = \kappa_{k+1,n}$ since $\omega \in \mathbf{V}_k$, while letting $v = \omega$ yields $\omega \equiv 0$. So, (5.2) holds with $\tau_{k+1,n} = \frac{1}{\kappa_{k+1,n}} > 0$. Just as theorem 5.1, $\tau_{k+1,n}$ is the least positive eigenvalue of such kind. □

Clearly, this process can be iterated to derive a countable, increasing sequence $\{\tau_{k,n}\}_{k=1}^{\infty}$ of regularized harmonic Steklov eigenvalues such that

Theorem 5.4. *We have* $\lim_{k\to\infty} \tau_{k,n} = \infty$ *for all* $n = 1, 2, \ldots$.

Proof. Since the family $\{\mathfrak{u}_{k,n}\}_{k=1}^{\infty}$ of associated regularized harmonic Steklov eigenfunctions is part of an orthonormal basis for $H^1_{\hat{0}}(U_n)$ with respect to $\langle \cdot, \cdot \rangle_{H^1(U_n)}$, it converges weakly to zero. Therefore, the *compact trace theorem* on U_n yields a subsequence $\{\mathfrak{u}_{k_l,n}\}_{l=1}^{\infty}$ such that $\gamma\mathfrak{u}_{k_l,n} \to 0$ in $L^2(\partial U, d\sigma)$ as $l \to \infty$, from which we have $\kappa_{k_l,n} = \mathcal{T}(\mathfrak{u}_{k_l,n}) \to 0$, and thereby $\tau_{k_l,n} \to \infty$, as $l \to \infty$. $\square$

From (5.2), we see that, for every $k = 1, 2, \ldots$,

$$\int_{U_n} (\nabla\mathfrak{u}_{k,n} \cdot \nabla v + \mathfrak{u}_{k,n} v)\, dx = 0, \quad \forall\, v \in C_c^1(U_n)\,. \tag{5.14}$$

Let $\mathscr{N}_{\hat{0},\mathcal{L}_1}(U_n)$ be the null space of the operator $\mathcal{L}_1$, given by (5.1), in $H^1_{\hat{0}}(U_n)$. That is, it is the collection of all functions in $H^1_{\hat{0}}(U_n)$ such that (5.14) is satisfied. $\mathscr{N}_{\hat{0},\mathcal{L}_1}(U_n)$ is called the subspace of regularized harmonic functions in $H^1_{\hat{0}}(U_n)$. We can accordingly decompose the space $H^1_{\hat{0}}(U_n)$ such as

$$H^1_{\hat{0}}(U_n) = \mathscr{N}_{\hat{0},\mathcal{L}_1}(U_n) \oplus_{H^1(U_n)} H^1_0(U_n)\,. \tag{5.15}$$

Theorem 5.5. *The family $\{\mathfrak{u}_{k,n}\}_{k=1}^{\infty}$ of regularized harmonic Steklov eigenfunctions is a maximal $\langle\cdot,\cdot\rangle_{H^1(U_n)}$-orthonormal subset of $\mathscr{N}_{\hat{0},\mathcal{L}_1}(U_n)$.*

Proof. Obviously, by definition, these eigenfunctions $\{\mathfrak{u}_{k,n}\}_{k=1}^{\infty}$ are in $\mathscr{N}_{\hat{0},\mathcal{L}_1}(U_n)$, and are $\langle\cdot,\cdot\rangle_{H^1(U_n)}$-orthonormal. Also, one has $\lim_{k\to\infty} \mathcal{T}(\mathfrak{u}_{k,n}) = 0$. If they are not maximal in $\mathscr{N}_{\hat{0},\mathcal{L}_1}(U_n)$, there would exist some function $\mathfrak{u}_n \in \mathscr{N}_{\hat{0},\mathcal{L}_1}(U_n)$ such that $||\mathfrak{u}_n||_{H^1(U_n)} = 1$ and $\langle\mathfrak{u}_n, \mathfrak{u}_{k,n}\rangle_{H^1(U_n)} = 0$ for all $k = 1, 2, \ldots$. If $\mathcal{T}(\mathfrak{u}_n) > 0$, a $k_0 \in \mathbb{N}$ can be found such that $\kappa_{k_0,n} \geq \mathcal{T}(\mathfrak{u}_n) > \kappa_{k_0+1,n}$. However, by definition of $\mathfrak{u}_{k_0+1,n}$, $\mathcal{T}(\mathfrak{u}_n) \leq \kappa_{k_0+1,n}$ must be true, for $\mathfrak{u}_n$ is in $\mathbf{K}_{k_0}$. Nevertheless, $\mathcal{T}(\mathfrak{u}_n) = 0$ yields $\mathfrak{u}_n \in H^1_0(U_n)$, so that $\mathfrak{u}_n \equiv 0$ as it is in $\mathscr{N}_{\hat{0},\mathcal{L}_1}(U_n) \cap H^1_0(U_n)$. $\square$

Thus, $\{\mathfrak{u}_{k,n}\}_{k=1}^{\infty}$ provides a $\langle\cdot,\cdot\rangle_{H^1(U_n)}$-orthonormal basis for the space $\mathscr{N}_{\hat{0},\mathcal{L}_1}(U_n)$.

On the other hand, define

$$u_{k,n} := \sqrt{\tau_{k,n}}\mathfrak{u}_{k,n}. \tag{5.16}$$

From (5.2) and the condition that $||\mathfrak{u}_{k,n}||_{H^1(U_n)} = 1$, $||\gamma u_{k,n}||_{2,\partial U} = 1$ follows for every $k = 1, 2, \ldots$. We can consequently derive the result below.

Corollary 5.6. *The sequence $\{\gamma u_{k,n}\}_{k=1}^{\infty}$ of trace functions of $\{u_{k,n}\}_{k=1}^{\infty}$ provides a $\langle \cdot, \cdot \rangle_{2,\partial U}$-orthonormal basis for the space $L^2(\partial U, d\sigma)$.*

Proof. By (5.9) and (5.16), $\{\gamma u_{k,n}\}_{k=1}^{\infty}$ is orthonormal. Now, let $g \in L^2(\partial U, d\sigma) \subseteq H^{-\frac{1}{2}}(\partial U, d\sigma)$ be such that $\langle g, \gamma u_{k,n} \rangle_{2,\partial U} = 0$ for each $k = 1, 2, \ldots$. Consider the *mixed Dirichlet-Neumann type* boundary value problem such as

$$\begin{cases} u - \Delta u = 0 \text{ in } U_n, \text{ subject to} \\ \\ D_\nu u = g \text{ on } \partial U, \text{ and } \gamma u = 0 \text{ on } S_{r^n}. \end{cases} \tag{5.17}$$

The H^1-solvability of (5.17) in $H^1(U_n)$ is answered in Auchmuty [2, 6]. In particular, for such a g as given, we have a unique weak solution.

Let $u \in \mathscr{N}_{0,\mathcal{L}_1}(U_n)$ be this unique solution. As we already showed, $\{\mathfrak{u}_{k,n}\}_{k=1}^{\infty}$ is a $\langle \cdot, \cdot \rangle_{H^1(U_n)}$-orthonormal basis for $\mathscr{N}_{0,\mathcal{L}_1}(U_n)$. Therefore, $u = \sum_{k=1}^{\infty} c_k \mathfrak{u}_{k,n}$, with c_k being constants for all $k = 1, 2, \ldots$. Then, on ∂U,

$$g = D_\nu u = \sum_{k=1}^{\infty} c_k D_\nu \mathfrak{u}_{k,n} = \sum_{k=1}^{\infty} c_k \tau_{k,n} \frac{\gamma u_{k,n}}{\sqrt{\tau_{k,n}}}. \tag{5.18}$$

So, $c_k = \frac{1}{\sqrt{\tau_{k,n}}} \langle g, \gamma u_{k,n} \rangle_{2,\partial U} = 0$ for each $k \geq 1$, and thus $g \equiv 0$. $\square$

5.2 The Harmonic Case

In this section, analogous to (5.1), we consider the weak solvability of the following *mixed Dirichlet-Robin type* eigenvalue problem in $H^1(U_n)$

$$\begin{cases} -\Delta u = 0 \text{ in } U_n, \text{ subject to} \\ \\ D_\nu u = \delta u \text{ on } \partial U, \text{ and } \gamma u = 0 \text{ on } S_{r^n}. \end{cases} \tag{5.19}$$

Systems like (5.19) are called the *harmonic Steklov eigenproblems*, so that ours are to find the non-trivial solutions in $H^1_{\hat{0}}(U_n) \times (0, \infty)$ such that

$$\int_{U_n} \nabla u \cdot \nabla v \, dx - \delta \int_{\partial U} \gamma u \cdot \gamma v \, d\sigma = 0, \quad \forall\, v \in H^1_{\hat{0}}(U_n). \tag{5.20}$$

Let $\mathbf{B}$ be the closed unit ball of $H^1_{\hat{0}}(U_n)$ with respect to $||\cdot||_{\partial, U_n}$. That is,

$$\mathbf{B} := \left\{ u \in H^1_{\hat{0}}(U_n) : \int_{U_n} |\nabla u|^2 \, dx + \int_{\partial U} (\gamma u)^2 \, d\sigma \leq 1 \right\}. \tag{5.21}$$

Consider the variational principle $(\mathfrak{H}\mathfrak{S}_{1,n})$ of maximizing $\mathcal{T}$ on $\mathbf{B}$, and write

$$\beta_{1,n} := \sup_{u \in \mathbf{B}} \mathcal{T}(u). \tag{5.22}$$

As $||\cdot||_{H^1(U_n)}$ and $||\cdot||_{\partial, U_n}$ are equivalent on $H^1(U_n)$, and as $H^1_{\hat{0}}(U_n)$ is maximal in $H^1(U_n)$ with zero traces on S_{r^n}, $\mathbf{B}$ is bounded, closed and convex in $H^1_{\hat{0}}(U_n)$, which enables us to prove the existence result for solutions of $(\mathfrak{H}\mathfrak{S}_{1,n})$.

Theorem 5.7. *There exists some maximizer $\mathfrak{s}_{1,n}$ (so does $-\mathfrak{s}_{1,n}$) of $\mathcal{T}$ in $\mathbf{B}$ such* *that* $||\mathfrak{s}_{1,n}||_{\partial,U_n} = 1$*, which is an eigenfunction of the problem (5.20) corresponding to the least positive eigenvalue* $\delta_{1,n}$ *with* $\delta_{1,n} := \frac{1-\beta_{1,n}}{\beta_{1,n}}$.

Proof. As $\mathbf{B}$ is a bounded, closed, convex subset, it is weakly compact in $H^1_{\hat{0}}(U_n)$. Thus, $\mathcal{T}$ attains its supremum, as a maximum, at a function $\mathfrak{s}_{1,n} \in \mathbf{B}$ such that $\beta_{1,n} = \mathcal{T}(\mathfrak{s}_{1,n}) > 0$ is finite. Similarly, $||\mathfrak{s}_{1,n}||_{\partial,U_n} = 1$ holds. Now, if $\beta_{1,n} = 1$, then $\int_{U_n} |\nabla \mathfrak{s}_{1,n}|^2 \, dx = 0$ by (5.21), so that $\mathfrak{s}_{1,n}$ is a constant. However, it is impossible, as one would have $\gamma \mathfrak{s}_{1,n} = \pm \frac{1}{\sqrt{\sigma(\partial U)}} \neq 0$ on ∂U yet $\gamma \mathfrak{s}_{1,n} = 0$ on S_{r^n}.

A Lagrangian functional for the above variational principle $(\mathfrak{H}\mathfrak{S}_{1,n})$ is described by $\mathcal{F}_2 : H^1_{\hat{0}}(U_n) \times [0,\infty) \to \mathbb{R}$, which is defined as, for some constant $\lambda \geq 0$,

$$\mathcal{F}_2(u,\lambda) := \lambda \left(\int_{U_n} |\nabla u|^2 \, dx + \int_{\partial U} (\gamma u)^2 \, d\sigma - 1 \right) - \int_{\partial U} (\gamma u)^2 \, d\sigma. \tag{5.23}$$

Analogously, we can derive that, as a critical point, $\mathfrak{s}_{1,n}$ satisfies

$$\lambda \left(\int_{U_n} \nabla \mathfrak{s}_{1,n} \cdot \nabla v \, dx + \int_{\partial U} \gamma \mathfrak{s}_{1,n} \cdot \gamma v \, d\sigma \right) - \int_{\partial U} \gamma \mathfrak{s}_{1,n} \cdot \gamma v \, d\sigma = 0 \tag{5.24}$$

for all $v \in H^1_{\hat{0}}(U_n)$. Letting $v = \mathfrak{s}_{1,n}$ yields $\lambda = \beta_{1,n}$. As a result, (5.20) follows with $\delta_{1,n} = \frac{1-\beta_{1,n}}{\beta_{1,n}} > 0$.

Finally, $\delta_{1,n}$ can be shown the least positive eigenvalue of (5.20). $\square$

Remark 5.8. *Again, as an application, by homogeneity, one has that*

$$||\gamma u||_{2,\partial U} \leq \sqrt{\frac{\beta_{1,n}}{1-\beta_{1,n}}} \, ||\nabla u||_{L^2(U_n;\mathbb{R}^N)}, \quad \forall\, u \in H^1_{\hat{0}}(U_n). \tag{5.25}$$

This further implies that, for the function space $H^1_{\hat{0}}(U_n)$*, the gradient* L^2*-norm is equivalent to* $||\cdot||_{H^1(U_n)}$*, as* $||\cdot||_{H^1(U_n)} \leq C_{U_n} ||\cdot||_{\partial,U_n}$ *for some* $C_{U_n} > 0$.

Given the first k (≥ 1) harmonic Steklov eigenvalues $\{\delta_{1,n}, \ldots, \delta_{k,n}\}$ and a set $\{\mathfrak{s}_{1,n}, \ldots, \mathfrak{s}_{k,n}\}$ of associated $\langle \cdot, \cdot \rangle_{\partial, U_n}$-orthonormal eigenfunctions, we show how to derive the next pair $(\mathfrak{s}_{k+1,n}, \delta_{k+1,n}) \in H^1_{\widehat{0}}(U_n) \times (0, \infty)$ below.

For each $k \geq 1$, let

$$\mathbf{B}_k := \left\{ u \in \mathbf{B} : \langle \gamma u, \gamma \mathfrak{s}_{i,n} \rangle_{2,\partial U} = 0 \text{ for } i = 1, 2, \ldots, k \right\}. \tag{5.26}$$

As $0 < \delta_{1,n} \leq \delta_{2,n} \leq \cdots \leq \delta_{k,n}$, by (3.17) and (5.20), it is the same to set

$$\mathbf{B}_k := \left\{ u \in \mathbf{B} : \langle u, \mathfrak{s}_{i,n} \rangle_{\partial, U_n} = 0 \text{ for } i = 1, 2, \ldots, k \right\}. \tag{5.27}$$

Now, consider the variational principle $(\mathfrak{H}\mathfrak{S}_{k+1,n})$ of maximizing the functional $\mathcal{T}$ on the bounded, closed, convex subset $\mathbf{B}_k \subseteq \mathbf{B} \subseteq H^1_{\widehat{0}}(U_n)$, and write

$$\beta_{k+1,n} := \sup_{u \in \mathbf{B}_k} \mathcal{T}(u). \tag{5.28}$$

Theorem 5.9. *There are maximizers $\pm\mathfrak{s}_{k+1,n}$ of $\mathcal{T}$ on $\mathbf{B}_k$ with $||\mathfrak{s}_{k+1,n}||_{\partial, U_n} = 1$. These functions are eigenfunctions of our problem (5.20) associated with the eigenvalue $\delta_{k+1,n}$ such that $\delta_{k+1,n} := \frac{1-\beta_{k+1,n}}{\beta_{k+1,n}}$. Besides, $\delta_{k+1,n}$ is the smallest eigenvalue for our problem greater than or equal to $\delta_{k,n}$.*

Proof. Obviously, by (5.26), $\mathbf{B}_k$ is a bounded, closed, convex subset of $H^1_{\widehat{0}}(U_n)$. So, it is also weakly compact. Hence, $\mathcal{T}$ attains its supremum, as a maximum, at some function $\mathfrak{s}_{k+1,n} \in \mathbf{B}_k$ such that $\beta_{k+1,n} = \mathcal{T}(\mathfrak{s}_{k+1,n}) > 0$ is finite. $\beta_{k+1,n} \leq \beta_{k,n} < 1$ follows, and, analogous to theorem 5.1, $||\mathfrak{s}_{k+1,n}||_{\partial, U_n} = 1$ holds.

Let $\mathbf{S}_k$ be the linear space spanned by $\{\mathfrak{s}_{1,n}, \mathfrak{s}_{2,n}, \ldots, \mathfrak{s}_{k,n}\}$. Then, for a constant

$\lambda \geq 0$ and some function ω in $\mathbf{S}_k$, we similarly have

$$\begin{aligned}
&\int_{\partial U} \gamma \mathfrak{s}_{k+1,n} \cdot \gamma v \, d\sigma - \left(\int_{U_n} \nabla \omega \cdot \nabla v \, dx + \int_{\partial U} \gamma \omega \cdot \gamma v \, d\sigma \right) \\
&= \lambda \left(\int_{U_n} \nabla \mathfrak{s}_{k+1,n} \cdot \nabla v \, dx + \int_{\partial U} \gamma \mathfrak{s}_{k+1,n} \cdot \gamma v \, d\sigma \right), \quad \forall \, v \in H^1_{\hat{0}} \left(U_n \right).
\end{aligned} \tag{5.29}$$

Recall (5.26) and (5.27). Letting $v = \mathfrak{s}_{k+1,n}$ yields $\lambda = \beta_{k+1,n}$ as $\omega \in \mathbf{S}_k$, and letting $v = \omega$ yields $\omega \equiv 0$, so that (5.20) holds with $\delta_{k+1,n} = \frac{1-\beta_{k+1,n}}{\beta_{k+1,n}} > 0$. Besides, $\delta_{k+1,n}$ can also be shown the least positive eigenvalue of such kind. $\square$

Analogically, this process can again be iterated to derive a countable, increasing sequence $\{\delta_{k,n}\}_{k=1}^{\infty}$ of harmonic Steklov eigenvalues such that

Theorem 5.10. *One has* $\lim_{k \to \infty} \delta_{k,n} = \infty$ *for each* $n = 1, 2, \ldots$.

Proof. Since the family $\{\mathfrak{s}_{k,n}\}_{k=1}^{\infty}$ of associated eigenfunctions is part of a $\langle \cdot, \cdot \rangle_{\partial, U_n}$-orthonormal basis for $H^1_{\hat{0}}(U_n)$, and since $||\cdot||_{H^1(U_n)}$ and $||\cdot||_{\partial, U_n}$ are equivalent on $H^1(U_n)$, being aware of remark 5.8, the gradient L^2-norm is determinant. Then, the *compact trace theorem* on U_n yields the desired result. $\square$

From (5.20), we can easily see that, for all $v \in C^1_c(U_n)$,

$$\int_{U_n} \nabla \mathfrak{s}_{k,n} \cdot \nabla v \, dx = 0 \tag{5.30}$$

holds for this family $\{\mathfrak{s}_{k,n}\}_{k=1}^{\infty}$ of harmonic Steklov eigenfunctions.

Take $\mathscr{H}_{\hat{0}}(U_n)$ to be the collection of all functions in $H^1_{\hat{0}}(U_n)$ that satisfy (5.30), and call it the subspace of harmonic functions in $H^1_{\hat{0}}(U_n)$. Because $H^1_0(U_n)$ is also the closure of the set $C^1_c(U_n)$ with respect to the norm $||\cdot||_{\partial, U_n}$, we thus have that the function space $H^1_{\hat{0}}(U_n)$ can be decomposed as

$$H^1_{\hat{0}}(U_n) = \mathscr{H}_{\hat{0}}(U_n) \oplus_{\partial,U_n} H^1_0(U_n)\,. \tag{5.31}$$

Theorem 5.11. *The family $\{\mathfrak{s}_{k,n}\}_{k=1}^{\infty}$ of associated harmonic Steklov eigenfunctions is a maximal $\langle\cdot,\cdot\rangle_{\partial,U_n}$-orthonormal subset of $\mathscr{H}_{\hat{0}}(U_n)$.*

Proof. Obviously, by definition, these eigenfunctions $\{\mathfrak{s}_{k,n}\}_{k=1}^{\infty}$ are in $\mathscr{H}_{\hat{0}}(U_n)$, and are $\langle\cdot,\cdot\rangle_{\partial,U_n}$-orthonormal. Moreover, $\lim_{k\to\infty}\beta_{k,n} = \lim_{k\to\infty}\mathcal{T}(\mathfrak{s}_{k,n}) = 0$. Therefore, we can similarly prove this result as discussed in theorem 5.5. □

This result actually says that $\mathscr{H}_{\hat{0}}(U_n)$ is exactly the closed subspace of $H^1_{\hat{0}}(U_n)$ generated by all these harmonic Steklov eigenfunctions $\{\mathfrak{s}_{k,n}\}_{k=1}^{\infty}$ with respect to the inner product $\langle\cdot,\cdot\rangle_{\partial,U_n}$. On the other hand, write

$$s_{k,n} := \sqrt{1+\delta_{k,n}}\,\mathfrak{s}_{k,n}. \tag{5.32}$$

From (5.20) and the condition that $||\mathfrak{s}_{k,n}||_{\partial,U_n} = 1$, $||\gamma s_{k,n}||_{2,\partial U} = 1$ follows for every $k = 1, 2, \ldots$. We can then derive the result below.

Corollary 5.12. *The sequence $\{\gamma s_{k,n}\}_{k=1}^{\infty}$ of trace functions of $\{s_{k,n}\}_{k=1}^{\infty}$ provides a $\langle\cdot,\cdot\rangle_{2,\partial U}$-orthonormal basis for the space $L^2(\partial U, d\sigma)$.*

Before to start our proof, we first discuss the H^1-solvability of

$$\begin{cases} -\Delta u = 0 \text{ in } U_n, \text{ subject to} \\ \\ D_\nu u = g \text{ on } \partial U, \text{ and } \gamma u = 0 \text{ on } S_{r^n}, \end{cases} \tag{5.33}$$

i.e., a *mixed Dirichlet-Neumann type* boundary value problem, in $H^1(U_n)$.

Without loss of generality, we consider both the harmonic case (5.33) and the regularized harmonic case (5.17) altogether. As a matter of fact, there will exist a variational principle for studying this. For $g \in L^2(\partial U, d\sigma)$, consider the problem of minimizing the functional $\mathcal{F} : H^1_{\hat{0}}(U_n) \to (-\infty, \infty]$, defined by

$$\mathcal{F}(u) := \int_{U_n} \left(\vartheta u^2 + |\nabla u|^2\right) dx - 2\int_{\partial U} g \cdot \gamma u \, d\sigma, \tag{5.34}$$

where ϑ is an index which equals either 0 or 1 such that $\vartheta = 0$ corresponds to the situation (5.33) while $\vartheta = 1$ corresponds to the situation (5.17).

Recalling remark 5.8 and applying *Hölder*'s *inequality*, we have

$$\left|\int_{\partial U} g \cdot \gamma u \, d\sigma\right| \leq ||g||_{2,\partial U} \cdot ||\gamma u||_{2,\partial U} \leq C_g \, ||\nabla u||_{L^2(U_n;\mathbb{R}^N)} \tag{5.35}$$

for all $u \in H^1_{\hat{0}}(U_n)$. So, $\mathcal{F}$ is coercive, strictly convex and continuous on $H^1_{\hat{0}}(U_n)$. Thus, $\mathcal{F}$ has at least one minimizer, say, $\mathfrak{m}$, which is a critical point of it in $H^1_{\hat{0}}(U_n)$, that is, $\mathcal{F}'(\mathfrak{m})(v) = 0$ for all $v \in H^1_{\hat{0}}(U_n)$. In consequence, one has

$$\int_{U_n} (\nabla \mathfrak{m} \cdot \nabla v + \vartheta \mathfrak{m} v) \, dx - \int_{\partial U} g \cdot \gamma v \, d\sigma = 0, \quad \forall \, v \in H^1_{\hat{0}}(U_n). \tag{5.36}$$

Therefore, the minimizer $\mathfrak{m} \in H^1_{\hat{0}}(U_n)$ satisfies weakly the system (5.17) in the case when $\vartheta = 1$ and the system (5.33) in the case when $\vartheta = 0$.

In addition, one sees that the existence of $\mathfrak{m}$ in $H^1_{\hat{0}}(U_n)$ actually is unique as the family of constant functions is excluded from this space.

Below, let's give the proof of corollary 5.12.

Proof. We see easily that $\{\gamma s_{k,n}\}_{k=1}^{\infty}$ is orthonormal in $L^2(\partial U, d\sigma)$, by (5.26) and (5.32). Let $g \in L^2(\partial U, d\sigma) \subseteq H^{-\frac{1}{2}}(\partial U, d\sigma)$ be such that $\langle g, \gamma s_{k,n}\rangle_{2,\partial U} = 0$ for all $k = 1, 2, \ldots$. Then, the system (5.33) admits a unique weak solution $u \in \mathscr{H}_{\hat{0}}(U_n) \subseteq H^1_{\hat{0}}(U_n)$. Write $\sum\limits_{k=1}^{\infty} c_k \mathfrak{s}_{k,n}$ as the series expansion of u in terms of $\{\mathfrak{s}_{k,n}\}_{k=1}^{\infty}$, where c_k is a constant for every $k = 1, 2, \ldots$. Hereby, on ∂U, one has

$$g = D_\nu u = \sum_{k=1}^{\infty} c_k D_\nu \mathfrak{s}_{k,n} = \sum_{k=1}^{\infty} c_k \delta_{k,n} \frac{\gamma s_{k,n}}{\sqrt{1+\delta_{k,n}}}. \tag{5.37}$$

Therefore, $c_k = \frac{\left\langle g, \gamma s_{k,n}\right\rangle_{2,\partial U}}{\delta_{k,n}\left(1+\delta_{k,n}\right)^{-\frac{1}{2}}} = 0$ for each $k \geq 1$, and thus $g \equiv 0$. $\square$

Chapter 6

The Exterior Steklov Eigenproblems

6.1 The Regularized Harmonic Case

In this part, we shall study the *exterior regularized harmonic Steklov eigenproblems*. That is, we want to find the non-trivial solutions (u, τ) in $H^1(U) \times (0, \infty)$ satisfying

$$\int_U (\nabla u \cdot \nabla v + uv)\, dx - \tau \int_{\partial U} \gamma u \cdot \gamma v \, d\sigma = 0, \quad \forall\, v \in H^1(U)\,. \tag{6.1}$$

Such eigenfunctions are weak solutions in $H^1(U)$ of

$$\begin{cases} \mathcal{L}_1(u) = u - \Delta u = 0 & \text{in } U, \\ \\ \text{subject to } D_\nu u = \tau u & \text{on } \partial U. \end{cases} \tag{6.2}$$

In view of (5.2) and (6.1), we derive constructively the non-trivial solutions for (6.1) as the limits of those for (5.2). We need to ensure $\tau_1 > 0$, as $\tau_{1,n}$ is decreasing with respect to n (see also Auchmuty [8] for another description).

Theorem 6.1. *There are non-trivial solutions $\pm u_1$ for (6.1) corresponding to the eigenvalue $\tau_1 > 0$.*

Proof. Upon zero extension to infinity, theorem 5.1 guarantees the existence of at least one solution $\mathfrak{u}_{1,n}$ of (6.1) in $E^1(U_n) \subseteq H^1(U)$, through (4.2), with $\tau_{1,n} > 0$, for each $n \geq 1$. In addition, when $n_1 \leq n_2$, then $\tau_{1,n_1} \geq \tau_{1,n_2}$ follows, as a consequence of the fact that $E^1(U_{n_1}) \subseteq E^1(U_{n_2})$ implies $\kappa_{1,n_1} \leq \kappa_{1,n_2}$ by definition.

For this class $\{\tau_{1,n}\}_{n=1}^{\infty}$ of eigenvalues and an associated set $\{\mathfrak{u}_{1,n}\}_{n=1}^{\infty}$ of eigenfunctions with $||\mathfrak{u}_{1,n}||_{H^1(U)} = 1$, recall (5.16) for $\left\{u_{1,n} = \sqrt{\tau_{1,n}}\,\mathfrak{u}_{1,n}\right\}_{n=1}^{\infty}$. Hence, we have $||\gamma u_{1,n}||_{2,\partial U}^2 = 1$ and $||u_{1,n}||_{H^1(U)}^2 = \tau_{1,n} \leq \tau_{1,1}$ for all $n = 1, 2, \ldots$.

There exists a subsequence $\left\{u_{1,n_j}\right\}_{j=1}^{\infty}$ of $\{u_{1,n}\}_{n=1}^{\infty}$ and a function u_1 in $H^1(U)$, as the function $u_{1,n}$'s are bounded with respect to $||\cdot||_{H^1(U)}$, such that

$$u_{1,n_j} \rightharpoonup u_1 \text{ in } H^1(U) \tag{6.3}$$

when $j \to \infty$, where $\rightharpoonup$ denotes the notion of weak convergence.

Set $\upsilon_{1,n_j} := \mathcal{X}_1 u_{1,n_j}$ (see (4.5)) for every $j = 1, 2, \ldots$. One then has

$$\begin{cases} \left|\left|\upsilon_{1,n_j}\right|\right|_{2,U_2} \leq \left|\left|u_{1,n_j}\right|\right|_{2,U_2} \leq \left|\left|u_{1,n_j}\right|\right|_{2,U} \leq \left|\left|u_{1,n_j}\right|\right|_{H^1(U)}, \\ \\ \left|\left|\nabla \upsilon_{1,n_j}\right|\right|_{L^2(U_2;\mathbb{R}^N)} \leq C\left|\left|u_{1,n_j}\right|\right|_{H^1(U_2)} \leq C\left|\left|u_{1,n_j}\right|\right|_{H^1(U)}, \end{cases} \tag{6.4}$$

where C is such a constant that $C \geq \sqrt{2} \max \left\{ 1, ||\nabla \mathcal{X}_1||_{\infty,U} \right\} > 0$.

Thus, $\left\{ \upsilon_{1,n_j} \right\}_{j=1}^{\infty}$ are in $H_{\circ}^1(U_2)$ by construction, when restricted on U_2. Besides, $\upsilon_1 := \mathcal{X}_1 u_1$ is also in $H_{\circ}^1(U_2)$. Moreover, from (6.3), as $j \to \infty$, we have

$$\upsilon_{1,n_j} \rightharpoonup \upsilon_1 \text{ in } H^1(U_2). \tag{6.5}$$

In fact, for all $\omega \in L^2(U_2)$ and each $i = 1, 2, \ldots, N$, it follows that

$$\begin{cases} \int_{U_2} \upsilon_{1,n_j} \omega \, dx = \int_U u_{1,n_j} (\mathcal{X}_1 \omega) \, dx \to \int_U u_1 (\mathcal{X}_1 \omega) \, dx = \int_{U_2} \upsilon_1 \omega \, dx, \\ \\ \int_{U_2} \left(D_i \upsilon_{1,n_j} \right) \omega \, dx = \int_U u_{1,n_j} (\omega D_i \mathcal{X}_1) \, dx + \int_U \left(D_i u_{1,n_j} \right) (\mathcal{X}_1 \omega) \, dx \\ \\ \to \int_U u_1 (\omega D_i \mathcal{X}_1) \, dx + \int_U (D_i u_1) (\mathcal{X}_1 \omega) \, dx = \int_{U_2} (D_i \upsilon_1) \, \omega \, dx \end{cases} \tag{6.6}$$

when $j \to \infty$, as $\mathcal{X}_1 \omega, \omega D_i \mathcal{X}_1 \in L^2(U)$ by zero extension to infinity.

From (6.4), $\left\{ \upsilon_{1,n_j} \right\}_{j=1}^{\infty}$ are bounded with respect to $||\cdot||_{H^1(U_2)}$, as $\{u_{1,n}\}_{n=1}^{\infty}$ are with respect to $||\cdot||_{H^1(U)}$. Since $\gamma \upsilon_1 = \gamma u_1$ and $\gamma \upsilon_{1,n_j} = \gamma u_{1,n_j}$ on ∂U for all $j \geq 1$, applying (6.5) and the *compact trace theorem* on U_2 to the function υ_{1,n_j}'s derives a subsequence $\left\{ \upsilon_{1,n_{j_l}} \right\}_{l=1}^{\infty}$ of $\left\{ \upsilon_{1,n_j} \right\}_{j=1}^{\infty}$ such that $\gamma u_{1,n_{j_l}} \to \gamma u_1$ in $L^2(\partial U, d\sigma)$ as $l \to \infty$. This further yields that $||\gamma u_1||_{2,\partial U} = \lim_{l \to \infty} \left|\left| \gamma u_{1,n_{j_l}} \right|\right|_{2,\partial U} = 1$.

On the other hand, the lower semicontinuity of norms implies that

$$||u_1||_{H^1(U)}^2 \leq \varliminf_{j \to \infty} \left|\left| u_{1,n_j} \right|\right|_{H^1(U)}^2 = \lim_{j \to \infty} \tau_{1,n_j} = \tau_1, \tag{6.7}$$

where $\tau_1 \geq 0$ is defined as the limit of the decreasing sequence $\{\tau_{1,n}\}_{n=1}^{\infty}$.

By definition, $\tau_1 = 0$ implies $\upsilon_1 \equiv 0$, which however contradicts the condition $||\gamma\upsilon_1||_{2,\partial U} = 1$ and the trace estimate on U_2. Thus, $\tau_1 > 0$ must be true.

Finally, substituting the pair $\left(u_{1,n_{j_l}}, \tau_{1,n_{j_l}}\right)$ into (5.2) and letting $l \to \infty$, then equation (6.1) follows from the approximation scheme proposition 4.1, (6.3), and the fact that $\gamma u_{1,n_{j_l}} \to \gamma u_1$ in $L^2(\partial U, d\sigma)$ when $l \to \infty$. □

Remark 6.2. *As $\tau_1 > 0$ holds,*

$$||\gamma u||_{2,\partial U} \leq \sqrt{\kappa_1}\, ||u||_{H^1(U)}, \quad \forall\, u \in H^1(U), \tag{6.8}$$

follows from the estimate (5.8) by taking limit, where $\kappa_1 := \frac{1}{\tau_1}$. This provides an explicit trace estimate upper bound (see also remark 3.4).

Now, given the first k (≥ 1) exterior regularized harmonic Steklov eigenvalues $\{\tau_1, \tau_2, \dots, \tau_k\}$ and an associated set $\{u_1, u_2, \dots, u_k\}$ of orthogonal Steklov eigenfunctions in $H^1(U)$ with respect to $\langle \cdot, \cdot \rangle_{H^1(U)}$, we shall describe below how to find the next eigenvalue τ_{k+1} and a corresponding eigenfunction u_{k+1}.

Via the proof of theorem 6.1, (6.1) and induction, we have

$$||u_j||^2_{H^1(U)} = \tau_j \text{ and } ||\gamma u_j||_{2,\partial U} = 1 \tag{6.9}$$

for $j = 1, 2, \dots, k$, while, for $j_1, j_2 = 1, 2, \dots, k$ with $j_1 \neq j_2$, we assume

$$\langle u_{j_1}, u_{j_2} \rangle_{H^1(U)} = \langle \gamma u_{j_1}, \gamma u_{j_2} \rangle_{2,\partial U} = 0. \tag{6.10}$$

Obviously, $0 < \tau_{1,n} \leq \cdots \leq \tau_{k,n} \leq \tau_{k+1,n}$ for all $n \geq 1$, which combined with the limiting process gives us $0 < \tau_1 \leq \cdots \leq \tau_k \leq \tau_{k+1} := \lim_{n\to\infty} \tau_{k+1,n}$.

Theorem 6.3. *There exists at least one non-trivial pair* (u_{k+1}, τ_{k+1}) *in* $H^1(U) \times (0, \infty)$ *for which equation (6.1) is satisfied, with* τ_{k+1} *being the eigenvalue, given as above, and* u_{k+1} *being an associated eigenfunction, such that*

$$||u_{k+1}||^2_{H^1(U)} = \tau_{k+1} \text{ and } ||\gamma u_{k+1}||_{2,\partial U} = 1, \tag{6.11}$$

and such that, for all $j = 1, 2, \ldots, k$,

$$\langle u_j, u_{k+1} \rangle_{H^1(U)} = \langle \gamma u_j, \gamma u_{k+1} \rangle_{2,\partial U} = 0. \tag{6.12}$$

Proof. The proof of theorem 6.1 ensures the existence of a subsequence $\left\{u_{1,n_{1,l}}\right\}_{l=1}^{\infty}$ of $\{u_{1,n}\}_{n=1}^{\infty}$ and a function u_1 such that $u_{1,n_{1,l}} \rightharpoonup u_1$ in $H^1(U)$ and $\gamma u_{1,n_{1,l}} \to \gamma u_1$ in $L^2(\partial U, d\sigma)$ as $l \to \infty$, and u_1 is an eigenfunction for (6.1) corresponds to $\tau_1 > 0$. Inductively, there is a subsequence $\left\{u_{k,n_{k,l}}\right\}_{l=1}^{\infty}$ of $\left\{u_{k,n_{k-1,l}}\right\}_{l=1}^{\infty}$ and a function u_k such that $u_{k,n_{k,l}} \rightharpoonup u_k$ in $H^1(U)$ and $\gamma u_{k,n_{k,l}} \to \gamma u_k$ in $L^2(\partial U, d\sigma)$ as $l \to \infty$, and such that (6.1) is satisfied by (u_k, τ_k) with $\tau_k > 0$. Accordingly, repeating the proof of theorem 6.1 derives again a subsequence $\left\{u_{k+1,n_{k+1,l}}\right\}_{l=1}^{\infty}$ of $\left\{u_{k+1,n_{k,l}}\right\}_{l=1}^{\infty}$ and a function u_{k+1} such that $u_{k+1,n_{k+1,l}} \rightharpoonup u_{k+1}$ in $H^1(U)$ and $\gamma u_{k+1,n_{k+1,l}} \to \gamma u_{k+1}$ in $L^2(\partial U, d\sigma)$ as $l \to \infty$, and such that equation (6.1) is satisfied by (u_{k+1}, τ_{k+1}), with $\tau_{k+1} \geq \tau_k > 0$ and $||\gamma u_{k+1}||_{2,\partial U} = \lim_{l\to\infty} \left|\left|\gamma u_{k+1,n_{k+1,l}}\right|\right|_{2,\partial U} = 1$. The existence of a non-trivial pair (u_{k+1}, τ_{k+1}) in $H^1(U) \times (0, \infty)$ for equation (6.1) is thus confirmed. Letting $u = v = u_{k+1}$ in (6.1) then yields $||u_{k+1}||^2_{H^1(U)} = \tau_{k+1}$.

Notice $\langle u_{k+1,n}, u_{j,n} \rangle_{H^1(U)} = \langle \gamma u_{k+1,n}, \gamma u_{j,n} \rangle_{2,\partial U} = 0$ by (5.9) and (5.10) while $||\gamma u_{j,n}||_{2,\partial U} = 1$ by (5.16) for all $n = 1, 2, \ldots$ and $j = 1, 2, \ldots, k$. Also, from (6.1), one has $\langle u, u_{k+1} \rangle_{H^1(U)} = 0$ if and only if $\langle \gamma u, \gamma u_{k+1} \rangle_{2,\partial U} = 0$.

Therefore, applying *Hölder*'s *inequality* derives that

$$\begin{aligned}\left|\langle \gamma u_j, \gamma u_{k+1}\rangle_{2,\partial U}\right| &\leq \left|\left\langle \gamma u_j - \gamma u_{j,n_{k+1,l}}, \gamma u_{k+1}\right\rangle_{2,\partial U}\right| \\ &\quad + \left|\left\langle \gamma u_{j,n_{k+1,l}}, \gamma u_{k+1} - \gamma u_{k+1,n_{k+1,l}}\right\rangle_{2,\partial U}\right| \\ &\leq \left|\left|\gamma u_{j,n_{k+1,l}} - \gamma u_j\right|\right|_{2,\partial U} \cdot \left|\left|\gamma u_{k+1}\right|\right|_{2,\partial U} \\ &\quad + \left|\left|\gamma u_{k+1,n_{k+1,l}} - \gamma u_{k+1}\right|\right|_{2,\partial U} \cdot \left|\left|\gamma u_{j,n_{k+1,l}}\right|\right|_{2,\partial U} \to 0\end{aligned} \tag{6.13}$$

as $l \to \infty$, so that (6.12) is indeed true for every $j = 1, 2, \ldots, k$. □

In the following, a property of the sequence of exterior regularized harmonic Steklov eigenvalues $\{\tau_k\}_{k=1}^{\infty}$, derived by iteration as shown above in the proofs of theorems 6.1 and 6.3, shall be described, a consequence of which further implies that each eigenvalue τ_k is at most of finite multiplicity for $k = 1, 2, \ldots$.

Theorem 6.4. *Under our hypothesis, it follows that*

$$\lim_{k\to\infty} \tau_k = \infty. \tag{6.14}$$

Proof. Suppose that, on the contrary, $\tau_\infty := \lim\limits_{k\to\infty} \tau_k < \infty$. From (6.9) or (6.11), $\left\{||u_k||_{H^1(U)}\right\}_{k=1}^{\infty}$ are uniformly bounded. So, there exists a subsequence $\left\{u_{k_j}\right\}_{j=1}^{\infty}$ of $\{u_k\}_{k=1}^{\infty}$ and a function u_∞ such that, when $j \to \infty$, $u_{k_j} \rightharpoonup u_\infty$ in $H^1(U)$ and $\gamma u_{k_j} \to \gamma u_\infty$ in $L^2(\partial U, d\sigma)$, exactly as shown in the proof of theorem 6.1. Since $||\gamma u_k||_{2,\partial U} = 1$ for $k = 1, 2, \ldots$, $||\gamma u_\infty||_{2,\partial U} = 1$ is also true. However, by (6.10) and (6.12), $\{\gamma u_k\}_{k=1}^{\infty}$ is part of an orthonormal basis for $L^2(\partial U, d\sigma)$, so that $\gamma u_k \rightharpoonup 0$. In consequence, $\gamma u_\infty \equiv 0$, and thus a contradiction follows. □

6.2 The Harmonic Case

In this section, we shall instead study the *exterior harmonic Steklov eigenproblems.* That is, we want to find the non-trivial solutions (s, δ) in $E^1(U) \times (0, \infty)$ satisfying

$$\int_U \nabla s \cdot \nabla v \, dx - \delta \int_{\partial U} \gamma s \cdot \gamma v \, d\sigma = 0, \quad \forall\, v \in E^1(U), \tag{6.15}$$

which can also be viewed as the weak form of the following system

$$-\Delta s = 0 \text{ in } U, \text{ subject to } D_\nu s = \delta s \text{ on } \partial U. \tag{6.16}$$

In view of (5.20) and (6.15), we can again derive constructively the non-trivial solutions of (6.15) as the limits of those of (5.20). The point is to have $\delta_1 > 0$.

Theorem 6.5. *There are non-trivial solutions $\pm s_1$ of (6.15) corresponding to the eigenvalue $\delta_1 > 0$.*

Proof. Similarly, theorem 5.7 ensures the existence of at least one solution $\mathfrak{s}_{1,n}$ of (6.15) in $E^1(U_n) \subseteq E^1(U)$ (see (4.9) now), with $\delta_{1,n} > 0$, for all $n \geq 1$.

Recall, for the eigenvalues $\{\delta_{1,n}\}_{n=1}^{\infty}$ and a set $\{\mathfrak{s}_{1,n}\}_{n=1}^{\infty}$ of associated eigenfunctions with $||\mathfrak{s}_{1,n}||_{\partial,U} = 1$, $\left\{s_{1,n} = \sqrt{1+\delta_{1,n}}\,\mathfrak{s}_{1,n}\right\}_{n=1}^{\infty}$ via (5.32). So, $||\gamma s_{1,n}||_{2,\partial U}^2 = 1$ and $||s_{1,n}||_{\partial,U}^2 = 1 + \delta_{1,n} \leq 1 + \delta_{1,1}$ for all $n = 1, 2, \ldots$. A subsequence $\{s_{1,n_l}\}_{l=1}^{\infty}$ of $\{s_{1,n}\}_{n=1}^{\infty}$ and a function s_1 in $E^1(U)$ exist such that, as $l \to \infty$,

$$\begin{cases} \nabla s_{1,n_l} \rightharpoonup \nabla s_1 & \text{in } U, \\ \\ \gamma s_{1,n_l} \rightharpoonup \gamma s_1 & \text{on } \partial U. \end{cases} \tag{6.17}$$

From proposition 4.2, and the fact that $E^1(U)$ is a Hilbert space and thereby contained in its own dual space, we let $l \to \infty$ in (5.20) to derive that

$$\int_U \nabla s_1 \cdot \nabla v \, dx - \delta_1 \int_{\partial U} \gamma s_1 \cdot \gamma v \, d\sigma = 0, \quad \forall\, v \in E^1(U), \tag{6.18}$$

where $\delta_1 \geq 0$ is the limit of the decreasing sequence $\{\delta_{1,n}\}_{n=1}^{\infty}$.

Below, we shall show $\delta_1 > 0$. Hence, s_1 will be an eigenfunction for the exterior harmonic Steklov eigenproblem (6.15) associated to the eigenvalue δ_1.

Now, for each $n = 1, 2, \ldots$, put $w_{1,n} := \begin{cases} \tilde{s}_{1,n} & \text{on } G, \\ s_{1,n} & \text{on } U. \end{cases}$ Here, $\tilde{s}_{1,n}$ is the interior harmonic extension of $s_{1,n}$ over G, as given by (3.20). Auchmuty (see [5, theorems 3.1 and 6.1]) shows that, using the sequence $\left\{\delta_j^G\right\}_{j=0}^{\infty}$ of interior harmonic Steklov eigenvalues, and an associated family $\left\{\mathfrak{s}_j^G\right\}_{j=0}^{\infty}$ of interior harmonic Steklov eigenfunctions on G that is a maximal $\langle \cdot, \cdot \rangle_{\partial, G}$-orthonormal subset of the space $\mathscr{H}(G)$ in $H^1(G)$ of all harmonic functions on G and whose trace functions on ∂U provide a complete orthogonal basis of $L^2(\partial U, d\sigma)$, the expansions below

$$\tilde{s}_{1,n} = \sum_{j=0}^{\infty} \left(1 + \delta_j^G\right) \left\langle \gamma s_{1,n}, \gamma \mathfrak{s}_j^G \right\rangle_{2,\partial U} \cdot \mathfrak{s}_j^G \tag{6.19}$$

hold for all $n = 1, 2, \ldots$, with $G = \mathbb{R}^N \setminus \overline{U}$, such that

$$\|\tilde{s}_{1,n}\|_{\partial,G}^2 = \sum_{j=0}^{\infty} \left(1 + \delta_j^G\right)^2 \left| \left\langle \gamma s_{1,n}, \gamma \mathfrak{s}_j^G \right\rangle_{2,\partial U} \right|^2 < \infty. \tag{6.20}$$

Note here, as discussed in Auchmuty [5, theorem 6.2], estimate (6.20) is a sufficient and necessary condition for saying that $\gamma \tilde{s}_{1,n} = \gamma s_{1,n} \in H^{\frac{1}{2}}(\partial U, d\sigma)$.

Next, we claim $\left\{||\tilde{s}_{1,n_l}||_{\partial,G}\right\}_{l=1}^{\infty}$ is bounded. On the contrary, suppose that there is a subsequence (for convenience still using) $\{\tilde{s}_{1,n_l}\}_{l=1}^{\infty}$ such that $||\tilde{s}_{1,n_l}||_{\partial,G} \to \infty$ as $l \to \infty$. Via (6.17) and recalling the fact that $\gamma\mathfrak{s}_j^G \in H^{\frac{1}{2}}(\partial U, d\sigma) \subseteq H^{-\frac{1}{2}}(\partial U, d\sigma)$, the dual space of $H^{\frac{1}{2}}(\partial U, d\sigma)$, we have, for every $j = 0, 1, 2, \ldots$,

$$\int_{\partial U} \gamma s_{1,n_l} \cdot \gamma\mathfrak{s}_j^G \, d\sigma \to \int_{\partial U} \gamma s_1 \cdot \gamma\mathfrak{s}_j^G \, d\sigma \tag{6.21}$$

as $l \to \infty$. Yet, this would imply $||\tilde{s}_1||_{\partial,G} = \infty$ by (6.20), with $\tilde{s}_1$ being the interior harmonic extension of s_1 over G, contradicting $\gamma\tilde{s}_1 = \gamma s_1 \in H^{\frac{1}{2}}(\partial U, d\sigma)$ (also see (3.11) and the discussions after (3.20)). Our claim is thus verified.

In consequence, the *compact trace theorem* on G induces a subsequence (again we shall use) $\{\tilde{s}_{1,n_l}\}_{l=1}^{\infty}$ and a function $\mathtt{t}$ in $L^2(\partial U, d\sigma)$, such that

$$\gamma s_{1,n_l} = \gamma\tilde{s}_{1,n_l} \to \mathtt{t} \tag{6.22}$$

in $L^2(\partial U, d\sigma)$ as $l \to \infty$. By (6.17), $\gamma s_1 = \mathtt{t}$ holds. So, $\gamma s_{1,n_l} \to \gamma s_1$ in $L^2(\partial U, d\sigma)$ as $l \to \infty$, which further gives us $||\gamma s_1||_{2,\partial U} = \lim_{l\to\infty} ||\gamma s_{1,n_l}||_{2,\partial U} = 1$.

Finally, the lower semicontinuity of norms implies that

$$1 \le ||s_1||_{\partial,U}^2 \le \lim_{l\to\infty} ||s_{1,n_l}||_{\partial,U}^2 = \lim_{l\to\infty} 1 + \delta_{1,n_l} = 1 + \delta_1. \tag{6.23}$$

If $\delta_1 = 0$, we would have $\int_U |\nabla s_1|^2 \, dx = 0$ by substituting $v = s_1$ into (6.18). Thus, $s_1 \equiv 0$ from condition **(A1)**. In consequence, one then sees $\tilde{s}_1 \equiv 0$, which however is impossible in view of the trace estimate on G, as now $||\gamma s_1||_{2,\partial U} = 1$. Therefore, $\delta_1 > 0$ must hold and then $||s_1||_{\partial,U}^2 = 1 + \delta_1$ by (3.17) and (6.18). $\square$

Remark 6.6. *Again, as $\delta_1 > 0$ holds, for $\varpi_1 := \frac{1}{\delta_1}$, we see that*

$$||\gamma u||_{2,\partial U} \leq \sqrt{\varpi_1}\, ||\nabla u||_{L^2(U;\mathbb{R}^N)}\,, \quad \forall\, u \in E^1(U)\,, \tag{6.24}$$

from (5.25), which further implies that, for the function space $E^1(U)$, the gradient $L^2(U;\mathbb{R}^N)$-norm is equivalent to $||\cdot||_{\partial,U}$ (in other words, $\beta_1 := \lim_{n\to\infty} \beta_{1,n} < 1$). So, $E^1(U)$ is pretty much like the space $H^1_\partial(\Omega)$ of all functions in $H^1(\Omega)$ having null integral averages on $\partial\Omega$ (see Auchmuty [2, corollary 6.14]), where Ω is a bounded region with a compact, Lipschitz boundary. On the other hand, this further confirms that $||\cdot||_{\partial,U} \leq \sqrt{1+\varpi_1}\, ||\cdot||_{H^1(U)}$ on $H^1(U)$ (see remarks 3.4 and 6.2).

Now, given the first k (≥ 1) exterior harmonic Steklov eigenvalues $\{\delta_1, \delta_2, \ldots, \delta_k\}$ and an associated set $\{s_1, s_2, \ldots, s_k\}$ of orthogonal exterior harmonic Steklov eigenfunctions in $E^1(U)$ with respect to $\langle\cdot,\cdot\rangle_{\partial,U}$, we assume that

$$||s_j||^2_{\partial,U} = 1 + \delta_j \text{ and } ||\gamma s_j||_{2,\partial U} = 1 \tag{6.25}$$

for $j = 1, 2, \ldots, k$, while, for $j_1, j_2 = 1, 2, \ldots, k$ with $j_1 \neq j_2$,

$$\langle s_{j_1}, s_{j_2}\rangle_{\partial,U} = \langle \gamma s_{j_1}, \gamma s_{j_2}\rangle_{2,\partial U} = 0. \tag{6.26}$$

Notice $0 < \delta_{1,n} \leq \cdots \leq \delta_{k,n} \leq \delta_{k+1,n}$ for all $n \geq 1$. Then, combining this with the limiting process derives that $0 < \delta_1 \leq \cdots \leq \delta_k \leq \delta_{k+1} := \lim_{n\to\infty} \delta_{k+1,n}$.

Repeating the proof of theorem 6.3 and using induction yield a pair (s_{k+1}, δ_{k+1}) in $E^1(U) \times (0,\infty)$ for which equation (6.15) is satisfied such that $||\gamma s_{k+1}||_{2,\partial U} = 1$. Letting $s = v = s_{k+1}$ in (6.15) then yields $||s_{k+1}||^2_{\partial,U} = 1 + \delta_{k+1}$ from (3.17). Recall

$\langle s_{k+1,n}, s_{j,n}\rangle_{\partial,U} = \langle \gamma s_{k+1,n}, \gamma s_{j,n}\rangle_{2,\partial U} = 0$ by (5.26) and (5.27) and $||\gamma s_{j,n}||_{2,\partial U} = 1$ by (5.32) for all $n = 1, 2, \ldots$ and $j = 1, 2, \ldots, k$. Also, $\langle s, s_{k+1}\rangle_{\partial,U} = 0$ if and only if $\langle \gamma s, \gamma s_{k+1}\rangle_{2,\partial U} = 0$ by (3.17) and (6.15). *Hölder*'s *inequality* yields

$$\begin{aligned}\left|\langle \gamma s_j, \gamma s_{k+1}\rangle_{2,\partial U}\right| \leq & \left|\left\langle \gamma s_j - \gamma s_{j,n_{k+1,l}}, \gamma s_{k+1}\right\rangle_{2,\partial U}\right| \\ & + \left|\left\langle \gamma s_{j,n_{k+1,l}}, \gamma s_{k+1} - \gamma s_{k+1,n_{k+1,l}}\right\rangle_{2,\partial U}\right| \\ \leq & \left|\left|\gamma s_{j,n_{k+1,l}} - \gamma s_j\right|\right|_{2,\partial U} \cdot ||\gamma s_{k+1}||_{2,\partial U} \\ & + \left|\left|\gamma s_{k+1,n_{k+1,l}} - \gamma s_{k+1}\right|\right|_{2,\partial U} \cdot \left|\left|\gamma s_{j,n_{k+1,l}}\right|\right|_{2,\partial U} \to 0\end{aligned} \tag{6.27}$$

as $l \to \infty$, so that (6.26) is indeed true for each $j = 1, 2, \ldots, k, k+1$.

Summing up the preceding discussions, we actually proved that

Theorem 6.7. *There is at least one non-trivial pair (s_{k+1}, δ_{k+1}) in $E^1(U) \times (0, \infty)$ for (6.15), with δ_{k+1} and s_{k+1} being the $(k+1)$-th eigenvalue and a corresponding eigenfunction, such that (6.25) and (6.26) hold for $j = 1, 2, \ldots, k, k+1$.*

Again, from theorems 6.4, 6.5 and 6.7, we can similarly show that

Theorem 6.8. *Under our hypothesis, it follows that*

$$\lim_{k\to\infty} \delta_k = \infty. \tag{6.28}$$

Remark 6.9. *It is somewhat intriguing to derive our theorems 6.4 and 6.8 directly from the conclusions of theorems 5.4 and 5.10, respectively. Yet, we perhaps cannot do so, because rigorous arguments for ensuring $\lim\limits_{k\to\infty}\lim\limits_{n\to\infty} \cdot = \lim\limits_{n\to\infty}\lim\limits_{k\to\infty} \cdot$ are missing. However, we can view these results the other way around, since, given $n = 1, 2, \ldots$, the estimates $\tau_{k,n} \geq \tau_k$ and $\delta_{k,n} \geq \delta_k$ hold for all $k = 1, 2, \ldots$.*

Chapter 7

Decomposition of Function Spaces

First, note here, as shown after remark 4.4, $E^1(U)$ is the closure of $C^1_\omega\left(\overline{U}\right)$ with respect to $||\cdot||_{\partial,U}$, while, via remarks 2.1 and 2.2, $H^1(U)$ is the closure of $C^1_\omega\left(\overline{U}\right)$ with respect to $||\cdot||_{H^1(U)}$, which together with remark 6.6 again lead to $H^1(U) \subseteq E^1(U)$. In addition, $E^1_0(U)$, the subspace of $E^1(U)$ of functions having zero traces on ∂U, is the closure of $C^1_c(U)$ with respect to $||\cdot||_{\partial,U}$, or simply with respect to the gradient $L^2\left(U;\mathbb{R}^N\right)$-norm, while $H^1_0(U)$ is the closure of $C^1_c(U)$ with respect to $||\cdot||_{H^1(U)}$, so that this further implies that $H^1_0(U) \subseteq E^1_0(U)$.

In the following, for all $k = 1, 2, \ldots$, write, respectively,

$$\mathfrak{u}_k := \frac{1}{\sqrt{\tau_k}} u_k \in H^1(U), \tag{7.1}$$

and

$$\mathfrak{s}_k := \frac{1}{\sqrt{1+\delta_k}} s_k \in E^1(U). \tag{7.2}$$

Therefore, $||\mathfrak{u}_k||_{H^1(U)} = 1$ and $||\mathfrak{s}_k||_{\partial,U} = 1$, from (6.9) and (6.25).

7.1 The E^1-situation

A function u in $E^1(U)$ is said to be a weak solution of

$$-\Delta u = 0 \text{ on } U, \tag{7.3}$$

provided that

$$\langle u, v\rangle_{\partial,U} = \int_U \nabla u \cdot \nabla v \, dx = 0, \quad \forall\, v \in C_c^1(U). \tag{7.4}$$

All such functions will be called *harmonic* on U.

Let $\mathscr{H}(U)$ be the subspace of $E^1(U)$ of all harmonic functions on U. It is clear that $\mathscr{H}(U)$ and $E_0^1(U)$, as subspaces of $E^1(U)$, are actually $\langle\cdot,\cdot\rangle_{\partial,U}$-orthogonal. Therefore, we have the following decomposition of $E^1(U)$ such as

$$E^1(U) = \mathscr{H}(U) \oplus_{\partial,U} E_0^1(U). \tag{7.5}$$

Obviously, the eigenfunctions $\{s_k\}_{k=1}^{\infty}$ are in $\mathscr{H}(U)$ by (6.15) and (7.4). Also, they are maximal in $\mathscr{H}(U)$, which implies that $\mathscr{H}(U)$ is generated by all these exterior harmonic Steklov eigenfunctions. Moreover, their trace functions $\{\gamma s_k\}_{k=1}^{\infty}$ provide a $\langle\cdot,\cdot\rangle_{2,\partial U}$-orthonormal basis for $L^2(\partial U, d\sigma)$.

Theorem 7.1. *Under conditions **(B1)** and **(B2)**, the sequence $\{\gamma s_k\}_{k=1}^{\infty}$ of trace functions provides a $\langle\cdot,\cdot\rangle_{2,\partial U}$-orthonormal basis for $L^2(\partial U, d\sigma)$.*

Proof. First, a subset of the index set $\{n = 1, 2, \ldots\}$, let's call it $\{\alpha = 1, 2, \ldots\}$, can be found such that, for all $k = 1, 2, \ldots$, $s_{k,\alpha} \rightharpoonup s_k$ in $E^1(U)$ and $\gamma s_{k,\alpha} \to \gamma s_k$

in $L^2(\partial U, d\sigma)$ when $\alpha \to \infty$. Actually, using the same notations,

$$
\begin{array}{lll}
s_{1,n_{1,l}} \rightharpoonup s_1 & \gamma s_{1,n_{1,l}} \to \gamma s_1 & \{n_{1,1}, n_{1,2}, \ldots\} \subseteq \{n = 1, 2, \ldots\} \\
s_{2,n_{2,l}} \rightharpoonup s_2 & \gamma s_{2,n_{2,l}} \to \gamma s_2 & \{n_{2,1}, n_{2,2}, \ldots\} \subseteq \{n_{1,1}, n_{1,2}, \ldots\} \\
\ldots & \ldots & \ldots \\
s_{k,n_{k,l}} \rightharpoonup s_k & \gamma s_{k,n_{k,l}} \to \gamma s_k & \{n_{k,1}, n_{k,2}, \ldots\} \subseteq \{n_{k-1,1}, n_{k-1,2}, \ldots\} \\
s_{k+1,n_{k+1,l}} \rightharpoonup s_{k+1} & \gamma s_{k+1,n_{k+1,l}} \to \gamma s_{k+1} & \{n_{k+1,1}, n_{k+1,2}, \ldots\} \subseteq \{n_{k,1}, n_{k,2}, \ldots\} \\
\ldots & \ldots & \ldots
\end{array}
$$

holds accordingly. Taking the diagonal sequence $\{n_{1,1}, n_{2,2}, \ldots, n_{k,k}, n_{k+1,k+1}, \ldots\}$ and renaming it $\{\alpha = 1, 2, \ldots\}$, we get our desired index subset.

The $\langle\cdot,\cdot\rangle_{2,\partial U}$-orthonormality of this sequence $\{\gamma s_k\}_{k=1}^{\infty}$ follows easily by (6.25), (6.26) and induction. On the other hand, for all $g \in L^2(\partial U, d\sigma)$, $\langle g, \gamma s_{k,\alpha}\rangle_{2,\partial U} \to 0$ when $k \to \infty$, as $\{\gamma s_{k,\alpha}\}_{k=1}^{\infty}$ provides a $\langle\cdot,\cdot\rangle_{2,\partial U}$-orthonormal basis of $L^2(\partial U, d\sigma)$ for every given $\alpha = 1, 2, \ldots$. In addition, g can be expressed as

$$g = \sum_{k=1}^{\infty} \langle g, \gamma s_{k,\alpha}\rangle_{2,\partial U} \cdot \gamma s_{k,\alpha} \tag{7.6}$$

for all $\alpha = 1, 2, \ldots$ such that

$$||g||_{2,\partial U} = \sqrt{\sum_{k=1}^{\infty} \left|\langle g, \gamma s_{k,\alpha}\rangle_{2,\partial U}\right|^2} < \infty, \tag{7.7}$$

which however does not depend on α.

Now, let $g \in L^2(\partial U, d\sigma)$ be such that $\langle g, \gamma s_k\rangle_{2,\partial U} = 0$ for each $k \geq 1$.

If, for each $\varepsilon > 0$, there exists an integer $K \in \mathbb{N}$ and a subset $\{\alpha_l\}_{l=1}^{\infty}$ of the index set $\{\alpha = 1, 2, \ldots\}$ such that, for all $l = 1, 2, \ldots$, we have

$$\sqrt{\sum_{k=K+1}^{\infty} \left|\langle g, \gamma s_{k,\alpha_l}\rangle_{2,\partial U}\right|^2} \leq \frac{\varepsilon}{\sqrt{2}}, \tag{7.8}$$

then, accordingly, there exists an integer $\mathfrak{a}_{\varepsilon,K} \in \mathbb{N}$ such that

$$\left|\langle g, \gamma s_{1,\alpha_l}\rangle_{2,\partial U}\right|, \ \left|\langle g, \gamma s_{2,\alpha_l}\rangle_{2,\partial U}\right|, \ \cdots, \ \left|\langle g, \gamma s_{K,\alpha_l}\rangle_{2,\partial U}\right| \leq \frac{\varepsilon}{\sqrt{2K}} \tag{7.9}$$

for all $\alpha_l \geq \mathfrak{a}_{\varepsilon,K}$, as $\gamma s_{k,\alpha} \to \gamma s_k$ when $\alpha \to \infty$ for each $k = 1, 2, \ldots$. Thus, it leads to $||g||_{2,\partial U} \leq \varepsilon$, so that actually $g \equiv 0$ via the arbitrariness of ε.

If not, there would be a fixed $\varepsilon_0 > 0$ such that, for all integers $K \in \mathbb{N}$ and all subsets of the index set $\{\alpha = 1, 2, \ldots\}$, a corresponding index from each one could be found that violates the estimate (7.8). In consequence, for $\{\alpha = 1, 2, \ldots\}$ itself, an index, say, $\alpha = 1$ (do renumbering if necessary), exists such that

$$\sqrt{\sum_{k=K}^{\infty} \left|\langle g, \gamma s_{k,\alpha}\rangle_{2,\partial U}\right|^2} \geq \varepsilon_0 > 0 \tag{7.10}$$

for $\alpha = 1$; again, for $\{\alpha = 2, 3, \ldots\}$, an index, say, $\alpha = 2$, exists such that (7.10) holds for $\alpha = 2$; continuing like this, (7.10) will finally hold for all these indices $\alpha = 1, 2, \ldots$, independent of K. Letting $K \to \infty$ thereby derives a contradiction against (7.7) accordingly, which finally finishes our proof. $\square$

Theorem 7.2. *The family* $\{\mathfrak{s}_k\}_{k=1}^{\infty}$ *of exterior harmonic Steklov eigenfunctions, given by (7.2), is a maximal* $\langle\cdot,\cdot\rangle_{\partial,U}$*-orthonormal subset of* $\mathscr{H}(U)$.

Proof. Obviously, the $\langle\cdot,\cdot\rangle_{\partial,U}$-orthonormality of $\{\mathfrak{s}_k\}_{k=1}^{\infty}$ follows from (6.25), (6.26) and (7.2) by induction. Suppose there exists a $\mathfrak{s}_{\mathrm{o}} \in \mathscr{H}(U)$ such that $||\mathfrak{s}_{\mathrm{o}}||_{\partial,U} = 1$ and such that $\langle \mathfrak{s}_{\mathrm{o}}, \mathfrak{s}_k\rangle_{\partial,U} = 0$ for every $k = 1, 2, \ldots$. As a result, from (3.17), (6.15) and again (7.2), $\langle \gamma\mathfrak{s}_{\mathrm{o}}, \gamma s_k\rangle_{2,\partial U} = 0$ for all $k = 1, 2, \ldots$, so that $\gamma\mathfrak{s}_{\mathrm{o}} \equiv 0$ on ∂U and thus $\mathfrak{s}_{\mathrm{o}} \in E_0^1(U)$. Therefore, $\mathfrak{s}_{\mathrm{o}} \in \mathscr{H}(U) \cap E_0^1(U)$, that is, $\mathfrak{s}_{\mathrm{o}} \equiv 0$. $\square$

This result can also be interpreted as saying that $\mathscr{H}(U)$ is the closed subspace of $E^1(U)$, with all these exterior harmonic Steklov eigenfunctions $\{\mathfrak{s}_k\}_{k=1}^{\infty}$ being a $\langle\cdot,\cdot\rangle_{\partial,U}$-orthonormal basis. So, for any function u in $E^1(U)$, its associated $\langle\cdot,\cdot\rangle_{\partial,U}$-orthogonal projection into $\mathscr{H}(U)$ is uniquely determined by

$$\mathcal{P}_{\mathscr{H}(U)}(u) := \sum_{k=1}^{\infty} \langle u, \mathfrak{s}_k\rangle_{\partial,U} \cdot \mathfrak{s}_k, \tag{7.11}$$

and the projection operator $\mathcal{P}_{\mathscr{H}(U)} : E^1(U) \to \mathscr{H}(U)$ is onto.

In particular, *Parseval*'s *theorem* tells us that every function $s \in \mathscr{H}(U)$ has a unique series expansion of the following form

$$s = \sum_{k=1}^{\infty} \langle s, \mathfrak{s}_k\rangle_{\partial,U} \cdot \mathfrak{s}_k \tag{7.12}$$

such that

$$||s||_{\partial,U} = \sqrt{\sum_{k=1}^{\infty} \left| \langle s, \mathfrak{s}_k \rangle_{\partial,U} \right|^2} < \infty. \tag{7.13}$$

Moreover, for all $u \in E^1(U)$, γu can be uniquely represented as

$$\gamma u = \gamma\left(\mathcal{P}_{\mathscr{H}(U)}(u)\right) = \sum_{k=1}^{\infty} \frac{\langle u, \mathfrak{s}_k \rangle_{\partial,U}}{\sqrt{1+\delta_k}} \cdot \gamma s_k \tag{7.14}$$

on ∂U, via the decomposition of $E^1(U)$ by (7.5) and (7.2), such that

$$||\gamma u||_{2,\partial U} = \sqrt{\sum_{k=1}^{\infty} \frac{\left| \langle u, \mathfrak{s}_k \rangle_{\partial,U} \right|^2}{1+\delta_k}} \leq \frac{\left|\left|\mathcal{P}_{\mathscr{H}(U)}(u)\right|\right|_{\partial,U}}{\sqrt{1+\delta_1}} \leq \frac{||u||_{\partial,U}}{\sqrt{1+\delta_1}} < \infty, \tag{7.15}$$

where the increasing property of the sequence $\{\delta_k\}_{k=1}^{\infty}$ was applied.

For each $M \in \mathbb{N}$, setting

$$u_M := \sum_{k=1}^{M} \langle u, \mathfrak{s}_k \rangle_{\partial,U} \cdot \mathfrak{s}_k \in \mathscr{H}(U) \tag{7.16}$$

on U, and arguing in an analogous manner as above, with the M-th *truncated trace mapping* $\gamma_M : E^1(U) \to L^2(\partial U, d\sigma)$, defined by

$$\gamma_M(u) := \sum_{k=1}^{M} \frac{\langle u, \mathfrak{s}_k \rangle_{\partial,U}}{\sqrt{1+\delta_k}} \cdot \gamma s_k, \tag{7.17}$$

we have the following spectral approximation estimate.

Proposition 7.3. *Under the above hypothesis, for all $u \in E^1(U)$, one has*

$$\|(\gamma - \gamma_M)(u)\|_{2,\partial U} \leq \sqrt{\sum_{k=M+1}^{\infty} \frac{\left|\langle u, \mathfrak{s}_k \rangle_{\partial, U}\right|^2}{1+\delta_k}} \leq \frac{\|u - u_M\|_{\partial, U}}{\sqrt{1+\delta_{M+1}}}, \tag{7.18}$$

and the operator norm of $\gamma - \gamma_M$ is exactly $\frac{1}{\sqrt{1+\delta_{M+1}}}$, upon letting $u = \mathfrak{s}_{M+1}$.

Obviously, in sight of theorem 6.8, $\frac{1}{\sqrt{1+\delta_{M+1}}}$ tends to zero as $M \to \infty$.

On the other hand, noticing (3.25) and (3.26), (7.5) further implies that

$$D^1\left(\mathbb{R}^N\right) = \left[E_0^1(U) \oplus_{\partial, U} \mathscr{H}(U)\right] \oplus_{\nabla} \left[H_0^1(G) \oplus_{\partial, G} \mathscr{H}(G)\right]. \tag{7.19}$$

Here, $D^1\left(\mathbb{R}^N\right)$ is the finite energy space on $\mathbb{R}^N$ when $N \geq 3$, as given in [24].

7.2 The H^1-situation

A function u in $H^1(U)$ is said to be a weak solution of

$$-\Delta u + \mu^2 u = 0 \text{ on } U, \tag{7.20}$$

with $\mu > 0$ being a constant, provided that

$$\langle u, v \rangle_{H^1_\mu(U)} := \int_U \left(\mu^2 uv + \nabla u \cdot \nabla v\right) dx = 0, \quad \forall\, v \in C_c^1(U). \tag{7.21}$$

All such functions will be called *μ-regularized harmonic* on U.

Here, the weighted H^1_μ-inner product is defined in exactly the same manner as

(7.21), just replacing U by any region Ω and then taking $u, v \in H^1(\Omega)$.

Let $\mathscr{N}_\mu(U)$ be the subspace of $H^1(U)$ of all μ-regularized harmonic functions on U. Clearly, as subspaces of $H^1(U)$, $\mathscr{N}_\mu(U)$ and $H_0^1(U)$ are $\langle\cdot,\cdot\rangle_{H_\mu^1(U)}$-orthogonal. Consequently, we have the following decomposition of $H^1(U)$ such as

$$H^1(U) = \mathscr{N}_\mu(U) \oplus_{H_\mu^1(U)} H_0^1(U). \tag{7.22}$$

Remark 7.4. *One has $H^1(U) \subsetneq E^1(U)$. Thus, unlike in the bounded case, on U, as $\mu \to 0$, the space $H^1(U)$, used for finding weak solutions to (7.20), will blow up to a strictly larger one $E^1(U)$, used for finding weak solutions to (7.3).*

Just as what we have already said, there is no significant difference if we simply consider the standard situation where $\mu = 1$. Now, it is obvious that $\{u_k\}_{k=1}^\infty$ are in $\mathscr{N}_1(U)$ by (6.1) and (7.21). Also, they are maximal in $\mathscr{N}_1(U)$ so that $\mathscr{N}_1(U)$ is exactly the closed subspace of $H^1(U)$ generated by all these exterior regularized harmonic Steklov eigenfunctions. In addition, their trace functions $\{\gamma u_k\}_{k=1}^\infty$ again provide a $\langle\cdot,\cdot\rangle_{2,\partial U}$-orthonormal basis for $L^2(\partial U, d\sigma)$.

Using parallel ideas, we can prove the following result.

Theorem 7.5. *Under conditions **(B1)** and **(B2)**, the sequence $\{\gamma u_k\}_{k=1}^\infty$ of trace functions of $\{u_k\}_{k=1}^\infty$ provides a $\langle\cdot,\cdot\rangle_{2,\partial U}$-orthonormal basis for $L^2(\partial U, d\sigma)$. Also, the family $\{\mathfrak{u}_k\}_{k=1}^\infty$ of exterior regularized harmonic Steklov eigenfunctions, defined in (7.1), is a maximal $\langle\cdot,\cdot\rangle_{H^1(U)}$-orthonormal subset of $\mathscr{N}_1(U)$.*

Similar to the E^1-situation, for any function $u \in H^1(U)$, its associated $\langle\cdot,\cdot\rangle_{H^1(U)}$-

orthogonal projection into $\mathscr{N}_1(U)$ is uniquely determined by

$$\mathcal{P}_{\mathscr{N}_1(U)}(u) := \sum_{k=1}^{\infty} \langle u, \mathfrak{u}_k \rangle_{H^1(U)} \cdot \mathfrak{u}_k, \tag{7.23}$$

and the projection operator $\mathcal{P}_{\mathscr{N}_1(U)} : H^1(U) \to \mathscr{N}_1(U)$ is onto.

Accordingly, the trace γu of u can be uniquely represented as

$$\gamma u = \gamma\left(\mathcal{P}_{\mathscr{N}_1(U)}(u)\right) = \sum_{k=1}^{\infty} \frac{\langle u, \mathfrak{u}_k \rangle_{H^1(U)}}{\sqrt{\tau_k}} \cdot \gamma u_k \tag{7.24}$$

on ∂U, via the decomposition of $H^1(U)$ by (7.22) and (7.1), such that

$$||\gamma u||_{2,\partial U} = \sqrt{\sum_{k=1}^{\infty} \frac{\left|\langle u, \mathfrak{u}_k \rangle_{H^1(U)}\right|^2}{\tau_k}} \leq \frac{\left|\left|\mathcal{P}_{\mathscr{N}_1(U)}(u)\right|\right|_{H^1(U)}}{\sqrt{\tau_1}} \leq \frac{||u||_{H^1(U)}}{\sqrt{\tau_1}} < \infty, \tag{7.25}$$

where the increasing property of the τ_k's was applied.

Setting

$$u_M := \sum_{k=1}^{M} \langle u, \mathfrak{u}_k \rangle_{H^1(U)} \cdot \mathfrak{u}_k \in \mathscr{N}_1(U) \tag{7.26}$$

on U, and arguing in an analogous manner as above, with the M-th *truncated trace mapping* $\gamma_M : H^1(U) \to L^2(\partial U, d\sigma)$, defined by

$$\gamma_M(u) := \sum_{k=1}^{M} \frac{\langle u, \mathfrak{u}_k \rangle_{H^1(U)}}{\sqrt{\tau_k}} \cdot \gamma u_k, \tag{7.27}$$

for $M = 1, 2, \ldots$, we have a similar spectral approximation estimate.

Proposition 7.6. *Under the above assumptions, for all $u \in H^1(U)$, one has*

$$||(\gamma - \gamma_M)(u)||_{2,\partial U} \leq \sqrt{\sum_{k=M+1}^{\infty} \frac{\left|\langle u, \mathfrak{u}_k \rangle_{H^1(U)}\right|^2}{\tau_k}} \leq \frac{||u - u_M||_{H^1(U)}}{\sqrt{\tau_{M+1}}}, \tag{7.28}$$

and the operator norm of $\gamma - \gamma_M$ is exactly $\frac{1}{\sqrt{\tau_{M+1}}}$, upon letting $u = \mathfrak{u}_{M+1}$.

Again, in view of theorem 6.4, $\frac{1}{\sqrt{\tau_{M+1}}}$ tends to zero as $M \to \infty$.

Finally, a result of Auchmuty (see [2, section 5]) implies that

$$H^1(G) = \mathscr{N}_1(G) \oplus_{H^1(G)} H_0^1(G), \tag{7.29}$$

where $\mathscr{N}_1(G)$ denotes the subspace of $H^1(G)$ of all regularized harmonic functions on $G = \mathbb{R}^N \setminus \overline{U}$. Combing this with (3.13) and (7.22), it follows that

$$H^1(\mathbb{R}^N) = \left[H_0^1(U) \oplus_{H^1(U)} \mathscr{N}_1(U)\right] \oplus_{H^1(\mathbb{R}^N)} \left[H_0^1(G) \oplus_{H^1(G)} \mathscr{N}_1(G)\right]. \tag{7.30}$$

Chapter 8

Several Examples

It is of interest to describe the interior and exterior Steklov eigenproblems for some standard regions in $\mathbb{R}^3$, the physically most important situation.

Let $G = B_1$ and then let $U = \mathbb{R}^3 \setminus \overline{B}_1$. Write $A_1 := U$.

Set $\tilde{\sigma} := \frac{1}{4\pi}\sigma$ in order that $\tilde{\sigma}(S_1) = 1$ for $S_1 := \partial B_1$. Moreover, all eigenfunctions described below will be of $L^2(S_1, d\tilde{\sigma})$-normalization.

Denote $Y_{j,l}(\theta, \phi)$ the (j,l)-th normalized *Laplace*'s *spherical harmonic*, derived from the j-th *Legendre polynomial* P_j and the (j,l)-th *associated Legendre function of the first kind* P_j^l. It is defined as, for $j \geq 0$ and $-j \leq l \leq j$,

$$Y_{j,l}(\theta, \phi) := \begin{cases} N_{(j,l)} P_j^{-l}(\cos\theta)\sin(-l\phi), & l = -1, -2, \ldots, -j, \\ \\ N_{(j,0)} P_j^0(\cos\theta) = N_{(j,0)} P_j(\cos\theta), & l = 0, \\ \\ N_{(j,l)} P_j^l(\cos\theta)\cos(l\phi), & l = 1, 2, \ldots, j. \end{cases} \tag{8.1}$$

Here, $N_{(j,l)}$ denotes the (j,l)-th normalization constant for j,l.

Given $j \geq 0$, $Y_{j,l}(\theta,\phi)$ are eigenfunctions of the *Laplace-Beltrami operator* Δ_∂ on S_1, associated with the eigenvalue $-j(j+1)$, for $-j \leq l \leq j$.

In consequence, in spherical polar coordinates $x = (r,\theta,\phi) \in \mathbb{R}^3$, with r being the radial distance, θ being the inclination (polar angle), and ϕ being the azimuthal angle, the interior harmonic Steklov eigenfunctions on B_1 are

$$s^{\mathfrak{i}}_{j,l} = r^j \cdot \sqrt{2} Y_{j,l}(\theta,\phi). \tag{8.2}$$

The interior harmonic Steklov eigenvalues are $\delta^{\mathfrak{i}}_j = 0,1,2,\ldots$, and the eigenvalue j has multiplicity exactly $2j+1$. In particular, when $j=0$, $s^{\mathfrak{i}}_{0,0} \equiv 1$.

As a consequence, we have that the family of interior harmonic Steklov eigenfunctions $\left\{\mathfrak{s}^{\mathfrak{i}}_{j,l}\right\}_{j=0,1,2,\ldots,-j\leq l\leq j}$ provides a $\langle\cdot,\cdot\rangle_{\partial,B_1}$-orthonormal basis for $\mathscr{H}(B_1)$, the subspace of $H^1(B_1)$ of all harmonic functions on B_1.

In addition, the exterior harmonic Steklov eigenfunctions on A_1 are

$$s^{\mathfrak{e}}_{k,l} = \frac{1}{r^k} \cdot \sqrt{2} Y_{k-1,l}(\theta,\phi) \tag{8.3}$$

for $k = 1,2,\ldots$ and $-k+1 \leq l \leq k-1$. The exterior harmonic Steklov eigenvalues are $\delta^{\mathfrak{e}}_k = 1,2,\ldots$, and the eigenvalue k has multiplicity exactly $2k-1$.

Let $E^1(A_1)$ be our finite energy space. The family of exterior harmonic Steklov eigenfunctions $\left\{\mathfrak{s}^{\mathfrak{e}}_{k,l}\right\}_{k=1,2,\ldots,-k+1\leq l\leq k-1}$ is a $\langle\cdot,\cdot\rangle_{\partial,A_1}$-orthonormal basis for $\mathscr{H}(A_1)$, the subspace of $E^1(A_1)$ of all harmonic functions on A_1. In particular, when $k=1$, $s^{\mathfrak{e}}_{1,0} = r^{-1}$, which is clearly not in $H^1(A_1)$ as it is not in $L^2(A_1)$. Besides, all these functions $r^{-\varsigma}\sin(r^{-\iota}-1)f(\theta,\phi)$, with $\iota > 0$ and $\varsigma \in \left(\frac{1}{2},\frac{3}{2}\right]$, are in $E^1_0(A_1)$ but clearly not in $H_0^1(A_1)$ in view of the conclusion of proposition 3.1.

On the other hand, the interior μ-regularized harmonic Steklov eigenvalues and associated interior μ-regularized harmonic Steklov eigenfunctions on B_1 are

$$\tau_j^{\mathrm{i}} = \frac{\mu \tilde{I}'_{j+\frac{1}{2}}(1)}{\tilde{I}_{j+\frac{1}{2}}(1)} = \frac{\sum_{t=0}^{\infty} \frac{(2t+j)\mu^{2t+j}}{4^t t! \Gamma\left(j+t+\frac{3}{2}\right)}}{\sum_{t=0}^{\infty} \frac{\mu^{2t+j}}{4^t t! \Gamma\left(j+t+\frac{3}{2}\right)}} \tag{8.4}$$

for $j = 0, 1, 2, \ldots$, and, with $-j \le l \le j$,

$$\begin{aligned} u_{j,l}^{\mathrm{i}} &= \tilde{I}_{j+\frac{1}{2}}(r) \cdot \sqrt{2} Y_{j,l}(\theta, \phi) \\ &= \frac{\sum_{t=0}^{\infty} \frac{\mu^{2t+j} r^{2t+j}}{4^t t! \Gamma\left(j+t+\frac{3}{2}\right)}}{\sum_{t=0}^{\infty} \frac{\mu^{2t+j}}{4^t t! \Gamma\left(j+t+\frac{3}{2}\right)}} \cdot \sqrt{2} Y_{j,l}(\theta, \phi). \end{aligned} \tag{8.5}$$

Here, $\tilde{I}_{j+\frac{1}{2}}(r) := \frac{\frac{1}{\sqrt{r}} I_{j+\frac{1}{2}}(r)}{I_{j+\frac{1}{2}}(1)}$ and $\sqrt{\frac{\pi}{2r}} I_{j+\frac{1}{2}}(r)$ denotes the j-th *modified spherical Bessel function of the first kind* for $j = 0, 1, 2, \ldots$.

In particular, when $j = 0$, $\tau_0^{\mathrm{i}} = \mu$ and $u_{0,0}^{\mathrm{i}} = \frac{e^{\mu r} - e^{-\mu r}}{r(e^{\mu} - e^{-\mu})}$.

Analogically, the family of interior μ-regularized harmonic Steklov eigenfunctions $\left\{\mathbf{u}_{j,l}^{\mathrm{i}}\right\}_{j=0,1,2,\ldots,-j\le l\le j}$ provides a $\langle \cdot, \cdot \rangle_{H_\mu^1(B_1)}$-orthonormal basis for $\mathscr{N}_\mu(B_1)$, the subspace of $H^1(B_1)$ of all μ-regularized harmonic functions on B_1.

As a matter of fact, given $j \ge 0$, one sees that $\sqrt{\frac{\pi}{2r}} I_{j+\frac{1}{2}}(r)$ is a solution of the following linear ordinary differential equation, when $r < 1$,

$$\left(\mu^2 + \frac{j(j+1)}{r^2}\right) g_j(r) - \frac{1}{r^2} \frac{d}{dr}\left(r^2 g_j(r)\right) = 0, \tag{8.6}$$

subject to $g_j(0) < \infty$. This then yields that $u_{j,l}^{\mathrm{i}}$ are μ-regularized harmonic Steklov eigenfunctions associated with the eigenvalue $\tau_j^{\mathfrak{i}}$ for $-j \leq l \leq j$.

In addition, on A_1, the exterior μ-regularized harmonic Steklov eigenvalues and associated exterior μ-regularized harmonic Steklov eigenfunctions are

$$\tau_k^{\mathfrak{e}} = -\frac{\mu \widetilde{K}'_{k-\frac{1}{2}}(1)}{\widetilde{K}_{k-\frac{1}{2}}(1)} = \frac{\sum_{t=0}^{k-1} \frac{(t+1+\mu)(k+t-1)!}{2^t t! \Gamma(k-t) \mu^{t+1}}}{\sum_{t=0}^{k-1} \frac{(k+t-1)!}{2^t t! \Gamma(k-t) \mu^{t+1}}} \tag{8.7}$$

for $k = 1, 2, \ldots$, and, with $-k+1 \leq l \leq k-1$,

$$\begin{aligned} u_{k,l}^{\mathfrak{e}} &= \widetilde{K}_{k-\frac{1}{2}}(r) \cdot \sqrt{2} Y_{k-1,l}(\theta, \phi) \\ &= e^{\mu(1-r)} \frac{\sum_{t=0}^{k-1} \frac{(k+t-1)!}{2^t t! \Gamma(k-t) \mu^{t+1} r^{t+1}}}{\sum_{t=0}^{k-1} \frac{(k+t-1)!}{2^t t! \Gamma(k-t) \mu^{t+1}}} \cdot \sqrt{2} Y_{k-1,l}(\theta, \phi), \end{aligned} \tag{8.8}$$

where $\widetilde{K}_{k-\frac{1}{2}}(r) := \frac{\frac{1}{\sqrt{r}} K_{k-\frac{1}{2}}(r)}{K_{k-\frac{1}{2}}(1)}$ and $\sqrt{\frac{\pi}{2r}} K_{k-\frac{1}{2}}(r)$ denotes the k-th *modified spherical Bessel function of the third kind* that solves equation (8.6) (replacing j by $k-1$), when $r > 1$ and subject to $\lim_{r \to \infty} g_k(r) < \infty$, for $k = 1, 2, \ldots$.

In particular, when $k = 1$, $\tau_1^{\mathfrak{e}} = 1 + \mu$ and $u_{1,0}^{\mathfrak{e}} = \frac{e^{\mu(1-r)}}{r}$. Even though $\frac{e^{\mu(1-r)}}{r}$ is in $H^1(A_1)$ when $\mu > 0$, one has $\lim_{\mu \to 0^+} \frac{e^{\mu(1-r)}}{r} = r^{-1} \in E^1(A_1) \setminus H^1(A_1)$.

Similarly, the family of exterior μ-regularized harmonic Steklov eigenfunctions $\left\{\mathfrak{u}_{k,l}^{\mathfrak{e}}\right\}_{k=1,2,\ldots,-k+1 \leq l \leq k-1}$ provides a $\langle \cdot, \cdot \rangle_{H_\mu^1(A_1)}$-orthonormal basis for $\mathscr{N}_\mu(A_1)$, the subspace of $H^1(A_1)$ of all μ-regularized harmonic functions on A_1.

For more details, see *M. Abramowitz and I.A. Stegun (eds), Handbook of Mathematical Functions with Formulas, Graphs, and Mathematical Tables, Chapter 10, National Bureau of Standards, Washington, DC, 1984.*

Chapter 9

Boundary Value Problems

9.1 The Harmonic Case

In this section, we shall describe the weak solvability in $E^1(U)$ of the harmonic boundary value problem, subject to various boundary conditions. Then, we shall discuss that briefly for the regularized harmonic case in $H^1(U)$.

First, let's consider this problem under Dirichlet data, *i.e.*,

$$-\Delta u = 0 \text{ in } U, \text{ subject to } \gamma u = \eta_1 \text{ on } \partial U, \tag{9.1}$$

with $\eta_1 \in L^2(\partial U, d\sigma)$. Suppose that (9.1) has a weak solution, say, $\mathfrak{h}_1$ in $E^1(U)$. By definition, $\mathfrak{h}_1 \in \mathscr{H}(U)$. Via the conclusion of theorem 7.2, we have

$$\mathfrak{h}_1 = \sum_{k=1}^{\infty} c_k \mathfrak{s}_k, \tag{9.2}$$

where the c_k's are constants for $k = 1, 2, \ldots$. Then, on ∂U, it follows that

$$\eta_1 = \gamma \mathfrak{h}_1 = \sum_{k=1}^{\infty} c_k \gamma \mathfrak{s}_k = \sum_{k=1}^{\infty} c_k \frac{\gamma s_k}{\sqrt{1+\delta_k}} \tag{9.3}$$

from (7.2). So, by theorem 7.1, one has $c_k = \sqrt{1+\delta_k}\, \langle \eta_1, \gamma s_k \rangle_{2,\partial U}$ for $k = 1, 2, \ldots$. Being aware of theorem 6.8, we see that $\mathfrak{h}_1 \in E^1(U)$ if and only if

$$\left(||\mathfrak{h}_1||^2_{\partial, U} = \right) \sum_{k=1}^{\infty} (1+\delta_k) \left| \langle \eta_1, \gamma s_k \rangle_{2,\partial U} \right|^2 < \infty. \tag{9.4}$$

Obviously, (9.4) need not be true for all $\eta_1 \in L^2(\partial U, d\sigma)$. In view of (3.11) and certain results below (3.20), (9.4) actually gives another definition of $H^{\frac{1}{2}}(\partial U, d\sigma)$, compared with the interior one by Auchmuty [4]. That is, as a proper subspace of $L^2(\partial U, d\sigma)$, a function $\eta_1 \in H^{\frac{1}{2}}(\partial U, d\sigma)$ whenever (9.4) holds.

Hence, (9.1) is weakly solvable in the space $E^1(U)$ if and only if (9.4) is satisfied, and its uniqueness follows from the first part of corollary 4.6.

Theorem 9.1. *Under the hypothesis, when subject to Dirichlet boundary condition $\eta_1 \in H^{\frac{1}{2}}(\partial U, d\sigma)$, the system (9.1) has a unique solution $\mathfrak{h}_1$ in $E^1(U)$, as described by (9.2), with $c_k = \sqrt{1+\delta_k}\, \langle \eta_1, \gamma s_k \rangle_{2,\partial U}$ for $k = 1, 2, \ldots$.*

Represent the exterior Poisson kernel, with $(x, y) \in (U, \partial U)$, by

$$\mathcal{P}^e_D(x, y) := \sum_{k=1}^{\infty} \sqrt{1+\delta_k}\, \mathfrak{s}_k(x) \cdot \gamma s_k(y). \tag{9.5}$$

When $\eta_1 \in H^{\frac{1}{2}}(\partial U, d\sigma)$, then $\mathcal{P}^e_D(\cdot, y)\eta_1(y)$ is integrable over ∂U, and, for $x \in U$, $\mathfrak{h}_1(x) = \int_{\partial U} \mathcal{P}^e_D(x, y)\eta_1(y)\, d\sigma$ is the unique weak solution of (9.1) in $E^1(U)$.

Keep in mind that, on $G = \mathbb{R}^N \setminus \overline{U}$, the constant functions are harmonic. Thus, (3.11) implies that, associated with these constant functions in $H^1(G)$, there is a family of harmonic functions in $E^1(U)$ whose trace functions on ∂U are constant. In consequence, one has that $\mathcal{P}^e_D(\cdot, y)$ is integrable over ∂U.

Analogously, via the sequence $\left\{\delta^G_j\right\}_{j=0}^{\infty}$ of interior harmonic Steklov eigenvalues and an associated family $\left\{\mathfrak{s}^G_j\right\}_{j=0}^{\infty}$ of interior harmonic Steklov eigenfunctions over G (see the proof of theorem 6.5), the representation of the interior Poisson kernel can be given by, for $(z, y) \in (G, \partial U)$ with $G = \mathbb{R}^N \setminus \overline{U}$,

$$\mathcal{P}_D(z, y) := \sum_{j=0}^{\infty} \sqrt{1 + \delta^G_j}\, \mathfrak{s}^G_j(z) \cdot \gamma s^G_j(y), \tag{9.6}$$

which can be compared to [13, theorem 5.33] (see also [2, section 9]), where

$$s^G_j := \sqrt{1 + \delta^G_j}\, \mathfrak{s}^G_j \tag{9.7}$$

so that $\left|\left|\gamma s^G_j\right|\right|_{2,\partial U} = 1$, as we know $\left|\left|\mathfrak{s}^G_j\right|\right|_{\partial, G} = 1$, for $j = 0, 1, 2, \ldots$.

In particular, when $N = 3$ and $G = B_1$, one has the identities

$$\begin{cases} \mathcal{P}_D(z, y) = \frac{1}{4\pi} \frac{1-|z|^2}{|z-y|^3} = \sum\limits_{j=0,1,2,\ldots,-j\le l\le j} 2\,|z|^j\, Y_{j,l}(\theta, \phi)\, Y_{j,l}(\theta^*, \phi^*) \\ \text{and} \\ \mathcal{P}^e_D(x, y) = \frac{1}{4\pi} \frac{|x|^2-1}{|x-y|^3} = \sum\limits_{k=1,2,\ldots,-k+1\le l\le k-1} \frac{2}{|x|^k} Y_{k-1,l}(\theta, \phi)\, Y_{k-1,l}(\theta^*, \phi^*) \end{cases} \tag{9.8}$$

for $x = (|x|, \theta, \phi) \in A_1 = \mathbb{R}^3 \setminus \overline{B}_1$, $z = (|z|, \theta, \phi) \in B_1$ and $y = (\theta^*, \phi^*) \in S_1$.

Moreover, for any data $g \in H^{\frac{1}{2}}(\partial U, d\sigma)$, there are two harmonic functions

$$\begin{cases} \mathfrak{h}_g(x) := \int_{\partial U} \mathcal{P}_D^e(x,y) g(y)\, d\sigma \in \mathscr{H}(U) \\ \text{and} \\ \tilde{\mathfrak{h}}_g(z) := \int_{\partial U} \mathcal{P}_D(z,y) g(y)\, d\sigma \in \mathscr{H}(G) \end{cases} \tag{9.9}$$

satisfying $\gamma\mathfrak{h}_g = \gamma\tilde{\mathfrak{h}}_g = g$ on ∂U, via (9.5) and (9.6), which differs from the classical double layer potential method (see Folland [19, chapter 3]).

As a matter of fact, our solution can be determined explicitly by its boundary data, involving the Steklov eigenvalues and eigenfunctions.

On the other hand, let's discuss the harmonic boundary value problem, subject to Neumann or Robin boundary conditions, *i.e.*,

$$-\Delta u = 0 \text{ in } U, \text{ subject to } D_\nu u + b\,(\gamma u) = \eta_2 \text{ on } \partial U, \tag{9.10}$$

where $b \geq 0$ is a constant, and $b = 0$ corresponds to the Neumann case.

Now, suppose that (9.10) has a weak solution in $E^1(U)$, say,

$$\mathfrak{h}_2 = \sum_{k=1}^{\infty} d_k \mathfrak{s}_k \tag{9.11}$$

for some constants $d_1, d_2, \ldots$. Then, on ∂U, it follows that, from (7.2),

$$\eta_2 = D_\nu \mathfrak{h}_2 + b\,(\gamma\mathfrak{h}_2) = \sum_{k=1}^{\infty} d_k\,(\delta_k + b)\,\frac{\gamma s_k}{\sqrt{1+\delta_k}}. \tag{9.12}$$

Thus, $d_k = \frac{\sqrt{1+\delta_k}}{\delta_k+b}\left\langle \eta_2, \gamma s_k\right\rangle_{2,\partial U}$ for $k=1,2,\ldots$. As $\frac{\sqrt{1+\delta_k}}{\delta_k+b} = O\left(\frac{1}{\sqrt{\delta_k}}\right)$,

$$\left(||\mathfrak{h}_2||^2_{\partial,U} =\right) \sum_{k=1}^{\infty}\left(\frac{1+\delta_k}{\left(\delta_k+b\right)^2}\left|\left\langle \eta_2, \gamma s_k\right\rangle_{2,\partial U}\right|^2\right) < \infty \tag{9.13}$$

whenever η_2 is in $L^2\left(\partial U, d\sigma\right)$, via the conclusion of theorem 6.8.

Then, (9.13) gives a definition for the dual space $H^{-\frac{1}{2}}\left(\partial U, d\sigma\right)$ of $H^{\frac{1}{2}}\left(\partial U, d\sigma\right)$. That is, $\eta_2 \in H^{-\frac{1}{2}}\left(\partial U, d\sigma\right)$ if and only if (9.13) is satisfied. Noticing ∂U is compact, $H^{-\frac{1}{2}}\left(\partial U, d\sigma\right)$ contains all these spaces $L^q\left(\partial U, d\sigma\right)$ for $q \geq \frac{2(N-1)}{N}$.

Hence, (9.10) is weakly solvable in the space $E^1\left(U\right)$ whenever (9.13) holds, and its uniqueness again follows from the first part of corollary 4.6.

Theorem 9.2. *Under our assumptions, and subject to Neumann or Robin boundary data $\eta_2 \in H^{-\frac{1}{2}}\left(\partial U, d\sigma\right)$, the system (9.10) has a unique solution $\mathfrak{h}_2$ in $E^1\left(U\right)$, as described by (9.11), with $d_k = \frac{\sqrt{1+\delta_k}}{\delta_k+b}\left\langle \eta_2, \gamma s_k\right\rangle_{2,\partial U}$ for $k=1,2,\ldots$.*

For the Neumann or Robin boundary value problems, we set

$$\mathcal{P}^e_{NR}(x,y) := \sum_{k=1}^{\infty}\frac{\sqrt{1+\delta_k}}{\delta_k+b}\,\mathfrak{s}_k(x)\cdot\gamma s_k(y) \tag{9.14}$$

to be the boundary solution operator ($\mathcal{NR}$-kernel) for some $b \geq 0$, with $b=0$ corresponding to the Neumann case, and thereby have $\mathfrak{h}_2(x) = \int_{\partial U}\mathcal{P}^e_{NR}(x,y)\eta_2(y)\,d\sigma$ is the unique weak solution of (9.10) in $E^1\left(U\right)$, when $\eta_2 \in H^{-\frac{1}{2}}\left(\partial U, d\sigma\right)$.

In view of (9.5) and theorem 6.8, $\mathcal{P}^e_{NR}(\cdot,y)$ is also integrable over ∂U. Moreover, $\mathcal{P}^e_{NR}(\cdot,y)\eta_2(y)$ is integrable over ∂U whenever $\eta_2 \in H^{-\frac{1}{2}}\left(\partial U, d\sigma\right)$.

In addition, series representations of the exterior and interior Neumann-Robin kernels can be described respectively such as $\mathcal{P}^e_{NR}(x,y)$ by (9.14) and

$$\mathcal{P}_{NR}(z,y) := \sum_{j=0}^{\infty} \frac{\sqrt{1+\delta_j^G}}{\delta_j^G + b}\, \mathfrak{s}_j^G(z) \cdot \gamma s_j^G(y), \tag{9.15}$$

so that

$$\begin{cases} \mathfrak{h}_h(x) := \int_{\partial U} \mathcal{P}^e_{NR}(x,y) h(y)\, d\sigma \in \mathscr{H}(U) \\ \text{and} \\ \tilde{\mathfrak{h}}_h(z) := \int_{\partial U} \mathcal{P}_{NR}(z,y) h(y)\, d\sigma \in \mathscr{H}(G) \end{cases} \tag{9.16}$$

are two harmonic functions obeying $D_\nu \mathfrak{h}_h + b\,(\gamma \mathfrak{h}_h) = D_\nu \tilde{\mathfrak{h}}_h + b\left(\gamma \tilde{\mathfrak{h}}_h\right) = h$ on ∂U, for any data $h \in H^{-\frac{1}{2}}(\partial U, d\sigma)$, which again is different from the well-known single layer potential method (see again Folland [19, chapter 3]).

Once more, using the Steklov eigenvalues and eigenfunctions, influences of the boundary data on our solution can be determined explicitly.

9.2 The Regularized Harmonic Case

In the regularized harmonic situation, for the system below

$$-\Delta u + u = 0 \text{ in } U, \text{ subject to } \gamma u = \eta_3 \text{ on } \partial U, \tag{9.17}$$

with $\eta_3 \in L^2(\partial U, d\sigma)$, it has a weak solution, say, $\mathfrak{k}_1 \in H^1(U)$ if and only if

$$\mathfrak{k}_1 = \sum_{k=1}^{\infty} \sqrt{\tau_k}\, \langle \eta_3, \gamma u_k \rangle_{2,\partial U} \cdot \mathfrak{u}_k, \tag{9.18}$$

via (7.1) and the conclusions of theorem 7.5, such that

$$\left(||\mathfrak{k}_1||^2_{H^1(U)}=\right)\sum_{k=1}^{\infty}\tau_k\left|\langle\eta_3,\gamma u_k\rangle_{2,\partial U}\right|^2<\infty, \tag{9.19}$$

which gives another description of $H^{\frac{1}{2}}(\partial U, d\sigma)$ by (3.11), (3.12) and theorem 6.4. Moreover, its uniqueness follows from the second part of corollary 4.6.

Theorem 9.3. *When subject to Dirichlet data $\eta_3 \in H^{\frac{1}{2}}(\partial U, d\sigma)$, the system (9.17) has a unique solution $\mathfrak{k}_1 = \sum\limits_{k=1}^{\infty}\sqrt{\tau_k}\,\langle\eta_3,\gamma u_k\rangle_{2,\partial U}\cdot\mathfrak{u}_k$ in $H^1(U)$.*

Let

$$\mathcal{K}^e_D(x,y) := \sum_{k=1}^{\infty}\sqrt{\tau_k}\mathfrak{u}_k(x)\cdot\gamma u_k(y) \tag{9.20}$$

be the boundary solution operator ($\mathcal{D}$-kernel). Then, $\mathfrak{k}_1(x) = \int_{\partial U}\mathcal{K}^e_D(x,y)\eta_3(y)\,d\sigma$ is the unique weak solution of (9.17) in $H^1(U)$, when $\eta_3 \in H^{\frac{1}{2}}(\partial U, d\sigma)$.

Analogously, using the identity $\gamma(H^1(U)) = \gamma(H^1(G))$ on ∂U instead, we have that $\mathcal{K}^e_D(\cdot,y)$ is an integrable function on ∂U. Further, $\mathcal{K}^e_D(\cdot,y)\eta_3(y)$ is integrable over ∂U whenever $\eta_3 \in H^{\frac{1}{2}}(\partial U, d\sigma)$.

On the other hand, for the system below

$$-\Delta u + u = 0 \text{ in } U,\ \text{subject to } D_\nu u + b(\gamma u) = \eta_4 \text{ on } \partial U, \tag{9.21}$$

with $b \geq 0$, it has a weak solution, say, $\mathfrak{k}_2 \in H^1(U)$ if and only if

$$\mathfrak{k}_2 = \sum_{k=1}^{\infty}\frac{\sqrt{\tau_k}}{\tau_k + b}\langle\eta_4,\gamma u_k\rangle_{2,\partial U}\cdot\mathfrak{u}_k \tag{9.22}$$

such that

$$\left(\|\mathfrak{k}_2\|^2_{H^1(U)} =\right) \sum_{k=1}^{\infty} \left(\frac{\tau_k}{(\tau_k + b)^2} \left| \langle \eta_4, \gamma u_k \rangle_{2,\partial U} \right|^2 \right) < \infty, \tag{9.23}$$

which yields another characterization of $H^{-\frac{1}{2}}(\partial U, d\sigma)$ via duality and theorem 6.4, and its uniqueness again follows from the second part of corollary 4.6.

Theorem 9.4. *When subject to Neumann or Robin data $\eta_4 \in H^{-\frac{1}{2}}(\partial U, d\sigma)$, the system (9.21) has a unique solution $\mathfrak{k}_2 = \sum_{k=1}^{\infty} \frac{\sqrt{\tau_k}}{\tau_k + b} \langle \eta_4, \gamma u_k \rangle_{2,\partial U} \cdot \mathfrak{u}_k$ in $H^1(U)$.*

Let

$$\mathcal{K}^e_{NR}(x, y) := \sum_{k=1}^{\infty} \frac{\sqrt{\tau_k}}{\tau_k + b} \mathfrak{u}_k(x) \cdot \gamma u_k(y) \tag{9.24}$$

be the boundary solution operator ($\mathcal{NR}$-kernel). So, $\mathfrak{k}_2(x) = \int_{\partial U} \mathcal{K}^e_{NR}(x, y)\eta_4(y)\, d\sigma$ is the unique weak solution of (9.21) in $H^1(U)$, when $\eta_4 \in H^{-\frac{1}{2}}(\partial U, d\sigma)$.

Noticing (9.20) and theorem 6.4, $\mathcal{K}^e_{NR}(\cdot, y)$ is also integrable over ∂U. Moreover, $\mathcal{K}^e_{NR}(\cdot, y)\eta_4(y)$ is integrable over ∂U whenever $\eta_4 \in H^{-\frac{1}{2}}(\partial U, d\sigma)$.

Remark 9.5. *Note here, the existence of weak solutions for (9.10) in $E^1(U)$ and these for (9.21) in $H^1(U)$ is guaranteed using variational arguments such as those shown in between (5.33) and (5.36), plus our remarks 6.2 and 6.6.*

Through the sequence $\left\{\tau^G_j\right\}_{j=0}^{\infty}$ of interior regularized harmonic Steklov eigenvalues, and a corresponding family $\left\{\mathfrak{u}^G_j\right\}_{j=0}^{\infty}$ of interior regularized harmonic Steklov eigenfunctions over G that consists of a maximal $\langle \cdot, \cdot \rangle_{H^1(G)}$-orthonormal subset of $\mathscr{N}_1(G)$ in $H^1(G)$ and whose $L^2(\partial U, d\sigma)$-normalized trace functions on ∂U,

$$u^G_j := \sqrt{\tau^G_j}\, \mathfrak{u}^G_j \text{ for } j = 0, 1, 2, \ldots, \tag{9.25}$$

provide a complete orthonormal basis for $L^2(\partial U, d\sigma)$ (see Auchmuty [2]), we define

$$\begin{cases} \mathcal{K}_D(z,y) := \sum_{j=0}^{\infty} \sqrt{\tau_j^G}\,\mathfrak{u}_j^G(z) \cdot \gamma u_j^G(y) \\ \text{and} \\ \mathcal{K}_{NR}(z,y) := \sum_{j=0}^{\infty} \frac{\sqrt{\tau_j^G}}{\tau_j^G + b}\mathfrak{u}_j^G(z) \cdot \gamma u_j^G(y). \end{cases} \tag{9.26}$$

In a similar manner, for any data $g \in H^{\frac{1}{2}}(\partial U, d\sigma)$ and $h \in H^{-\frac{1}{2}}(\partial U, d\sigma)$, there are two pairs of regularized harmonic functions, respectively,

$$\begin{cases} \mathfrak{k}_g(x) := \int_{\partial U} \mathcal{K}_D^e(x,y) g(y)\, d\sigma \in \mathscr{N}_1(U) \\ \text{and} \\ \tilde{\mathfrak{k}}_g(z) := \int_{\partial U} \mathcal{K}_D(z,y) g(y)\, d\sigma \in \mathscr{N}_1(G), \end{cases} \tag{9.27}$$

and

$$\begin{cases} \mathfrak{k}_h(x) := \int_{\partial U} \mathcal{K}_{NR}^e(x,y) h(y)\, d\sigma \in \mathscr{N}_1(U) \\ \text{and} \\ \tilde{\mathfrak{k}}_h(z) := \int_{\partial U} \mathcal{K}_{NR}(z,y) h(y)\, d\sigma \in \mathscr{N}_1(G), \end{cases} \tag{9.28}$$

such that $\gamma \mathfrak{k}_g = \gamma \tilde{\mathfrak{k}}_g = g$ and $D_\nu \mathfrak{k}_h + b(\gamma \mathfrak{k}_h) = D_\nu \tilde{\mathfrak{k}}_h + b\left(\gamma \tilde{\mathfrak{k}}_h\right) = h$ on ∂U.

Chapter 10

The Boundary $H^{\frac{1}{2}}$-Space

Recall that, using the interior harmonic Steklov eigenvalues $\left\{\delta_j^G\right\}_{j=0}^{\infty}$ and eigenfunctions $\left\{s_j^G\right\}_{j=0}^{\infty}$ (see (9.7)), Auchmuty [4] defines $H^{\frac{1}{2}}(\partial U, d\sigma)$, as a proper subspace of $L^2(\partial U, d\sigma)$, as the real Hilbert function space with respect to the inner product

$$\langle g, h\rangle_{H^{\frac{1}{2}}(\partial U)}^{G,\mathfrak{H}} := \sum_{j=0}^{\infty}\left(\sqrt{1+\delta_j^G}\int_{\partial U} g\gamma s_j^G\, d\sigma\right)\left(\sqrt{1+\delta_j^G}\int_{\partial U} h\gamma s_j^G\, d\sigma\right), \tag{10.1}$$

with the associated norm denoted $||g||_{H^{\frac{1}{2}}(\partial U)}^{G,\mathfrak{H}}$.

For all $g \in H^{\frac{1}{2}}(\partial U, d\sigma)$, let

$$\tilde{\mathfrak{h}}_g := \sum_{j=0}^{\infty}\sqrt{1+\delta_j^G}\left\langle g, \gamma s_j^G\right\rangle_{2,\partial U}\cdot \mathfrak{s}_j^G \tag{10.2}$$

be the unique *interior harmonic extension* of g over G. So, $\left|\left|\tilde{\mathfrak{h}}_g\right|\right|_{\partial,G} = ||g||_{H^{\frac{1}{2}}(\partial U)}^{G,\mathfrak{H}}$. As a result, it follows that the trace mapping

$$\gamma : \mathscr{H}(G) \to H^{\frac{1}{2}}(\partial U, d\sigma) \tag{10.3}$$

can be made an isometric isomorphism, via using the norms $||\cdot||_{\partial,G}$ and $||\cdot||^{G,\mathfrak{H}}_{H^{\frac{1}{2}}(\partial U)}$ on these two Hilbert spaces $\mathscr{H}(G)$ and $H^{\frac{1}{2}}(\partial U, d\sigma)$, respectively.

In this situation, denote the inverse of γ by $\mathcal{E}^{\mathfrak{H}}_{\mathfrak{i}}$. That is,

$$\mathcal{E}^{\mathfrak{H}}_{\mathfrak{i}} : H^{\frac{1}{2}}(\partial U, d\sigma) \to \mathscr{H}(G), \tag{10.4}$$

which again is an isometric isomorphism such that $\tilde{\mathfrak{h}}_g = \mathcal{E}^{\mathfrak{H}}_{\mathfrak{i}}(g)$.

Moreover, as the dual space of $H^{\frac{1}{2}}(\partial U, d\sigma)$ in terms of $\langle\cdot,\cdot\rangle_{2,\partial U}$, $H^{-\frac{1}{2}}(\partial U, d\sigma)$ can be defined as a real Hilbert function space with respect to the inner product

$$\langle g, h\rangle^{G,\mathfrak{H}}_{H^{-\frac{1}{2}}(\partial U)} := \sum_{j=0}^{\infty}\left(\frac{1}{\sqrt{1+\delta_j^G}}\int_{\partial U} g\gamma s_j^G\, d\sigma\right)\left(\frac{1}{\sqrt{1+\delta_j^G}}\int_{\partial U} h\gamma s_j^G\, d\sigma\right). \tag{10.5}$$

Accordingly, denote the generated norm by $||g||^{G,\mathfrak{H}}_{H^{-\frac{1}{2}}(\partial U)}$. Obviously, from this, one has, as function spaces, $H^{\frac{1}{2}}(\partial U, d\sigma) \subsetneq L^2(\partial U, d\sigma) \subsetneq H^{-\frac{1}{2}}(\partial U, d\sigma)$. We refer the reader to Auchmuty [4, 5] for more details on the preceding results.

Next, from the identities (3.11), (3.12), (7.5) and (7.22), one sees that, for all u in $\mathscr{H}(U)$, a function $\breve{u}$ in $H^1(U) \subsetneq E^1(U)$ can be found such that $\gamma\breve{u} = \gamma u \in H^{\frac{1}{2}}(\partial U, d\sigma)$ on ∂U. Since we assume that U is a 1-extension region (see remarks 2.1, 2.2 and (3.13)), via identity (3.25), there exists a unique function $\tilde{u}$ in $\mathscr{H}(G)$ such that $\gamma\tilde{u} = \gamma u \in H^{\frac{1}{2}}(\partial U, d\sigma)$ on ∂U, as described by (3.20). In consequence, a homomorphism from $\mathscr{H}(U)$ to $\mathscr{H}(G)$ exists, that is,

$$\mathcal{E}^{\mathfrak{H}}_{\mathfrak{i}} \circ \gamma : \mathscr{H}(U) \to \mathscr{H}(G). \tag{10.6}$$

On the other hand, for all $g \in H^{\frac{1}{2}}(\partial U, d\sigma)$, let

$$\mathfrak{h}_g := \sum_{k=1}^{\infty} \sqrt{1+\delta_k}\, \langle g, \gamma s_k \rangle_{2,\partial U} \cdot \mathfrak{s}_k \tag{10.7}$$

be the unique *exterior harmonic extension* of g over U, as given by (9.2).

Also, via the exterior harmonic Steklov eigenvalues $\{\delta_k\}_{k=1}^{\infty}$ and eigenfunctions $\{s_k\}_{k=1}^{\infty}$, $H^{\frac{1}{2}}(\partial U, d\sigma)$ is a real Hilbert function space with respect to

$$\langle g, h \rangle^{\mathfrak{H}}_{H^{\frac{1}{2}}(\partial U)} := \sum_{k=1}^{\infty} \left(\sqrt{1+\delta_k} \int_{\partial U} g \gamma s_k \, d\sigma \right) \left(\sqrt{1+\delta_k} \int_{\partial U} h \gamma s_k \, d\sigma \right), \tag{10.8}$$

and the associated norm is denoted $||g||^{\mathfrak{H}}_{H^{\frac{1}{2}}(\partial U)}$.

As a matter of fact, from theorems 7.1 and 7.2, one has $||\mathfrak{h}_g||_{\partial, U} = ||g||^{\mathfrak{H}}_{H^{\frac{1}{2}}(\partial U)}$. Therefore, any sequence $\{g_k\}_{k=1}^{\infty}$ of functions in $H^{\frac{1}{2}}(\partial U, d\sigma)$ is Cauchy with respect to $||\cdot||^{\mathfrak{H}}_{H^{\frac{1}{2}}(\partial U)}$, if and only if the generated sequence $\{\mathfrak{h}_{g_k}\}_{k=1}^{\infty}$ of harmonic functions in $\mathscr{H}(U)$ ($\subsetneq E^1(U)$), as defined by (10.7), is Cauchy with respect to $||\cdot||_{\partial, U}$. The completeness of the space $E^1(U)$, plus the identities (3.11) and (7.5), then gives us the corresponding completeness of the space $H^{\frac{1}{2}}(\partial U, d\sigma)$.

In addition, the foregoing argument also tells us that, for all $g \in H^{\frac{1}{2}}(\partial U, d\sigma)$, $||g||^{G,\mathfrak{H}}_{H^{\frac{1}{2}}(\partial U)} < \infty$ whenever $||g||^{\mathfrak{H}}_{H^{\frac{1}{2}}(\partial U)} < \infty$, by the homomorphism (10.6). Hence, a universal constant $C > 0$ can be found such that $||\cdot||^{G,\mathfrak{H}}_{H^{\frac{1}{2}}(\partial U)} \leq C\, ||\cdot||^{\mathfrak{H}}_{H^{\frac{1}{2}}(\partial U)}$. In fact, if not, there would be a sequence $\{g_k\}_{k=1}^{\infty}$ in $H^{\frac{1}{2}}(\partial U, d\sigma)$ such that $||g_k||^{\mathfrak{H}}_{H^{\frac{1}{2}}(\partial U)} = 1$ yet $||g_k||^{G,\mathfrak{H}}_{H^{\frac{1}{2}}(\partial U)} \geq k$. As now $||g_k||^{G,\mathfrak{H}}_{H^{\frac{1}{2}}(\partial U)}$ is bounded for each $k \geq 1$, we write $\eta_k := \frac{1}{||g_k||^{G,\mathfrak{H}}_{H^{\frac{1}{2}}(\partial U)}} g_k$ and have $||\eta_k||^{G,\mathfrak{H}}_{H^{\frac{1}{2}}(\partial U)} = 1$. Therefore, $\lim\limits_{k\to\infty} ||\eta_k||^{\mathfrak{H}}_{H^{\frac{1}{2}}(\partial U)} = 0$. Resorting to $\{\mathfrak{h}_{\eta_k}\}_{k=1}^{\infty}$, the completeness of $E^1(U)$ leads to $\lim\limits_{k\to\infty} \mathfrak{h}_{\eta_k} = 0$, so that $\lim\limits_{k\to\infty} \tilde{\mathfrak{h}}_{\eta_k} = 0$

via (10.6), too. This can not be true, for we would then have $\lim_{k\to\infty} ||\eta_k||^{G,\mathfrak{H}}_{H^{\frac{1}{2}}(\partial U)} = 0$ otherwise. Consequently, $||\cdot||^{G,\mathfrak{H}}_{H^{\frac{1}{2}}(\partial U)} \leq C\, ||\cdot||^{\mathfrak{H}}_{H^{\frac{1}{2}}(\partial U)}$ must hold. Finally, a standard result (see *e.g.* Brezis [15, corollary 2.8]) thus yields that the two norms $||\cdot||^{\mathfrak{H}}_{H^{\frac{1}{2}}(\partial U)}$ and $||\cdot||^{G,\mathfrak{H}}_{H^{\frac{1}{2}}(\partial U)}$ on the space $H^{\frac{1}{2}}(\partial U, d\sigma)$ are actually equivalent.

A corollary of this result says that the sequences of interior and exterior harmonic Steklov eigenvalues $\left\{\delta^G_j\right\}_{j=0}^{\infty}$ and $\{\delta_k\}_{k=1}^{\infty}$ will go to infinity at exactly the same speed, as $j, k \to \infty$. The preciseness is shown in chapter 8.

Furthermore, in this situation, the trace mapping

$$\gamma : \mathscr{H}(U) \to H^{\frac{1}{2}}(\partial U, d\sigma) \tag{10.9}$$

can again be made an isometric isomorphism, instead via using the norms $||\cdot||_{\partial,U}$ and $||\cdot||^{\mathfrak{H}}_{H^{\frac{1}{2}}(\partial U)}$ on $\mathscr{H}(U)$ and $H^{\frac{1}{2}}(\partial U, d\sigma)$, respectively.

Let $\mathcal{E}^{\mathfrak{H}}_{\mathfrak{e}}$ be its inverse. That is,

$$\mathcal{E}^{\mathfrak{H}}_{\mathfrak{e}} : H^{\frac{1}{2}}(\partial U, d\sigma) \to \mathscr{H}(U), \tag{10.10}$$

which again is an isometric isomorphism such that $\mathfrak{h}_g = \mathcal{E}^{\mathfrak{H}}_{\mathfrak{e}}(g)$.

Noticing (10.7) and the equivalence of the two norms $||\cdot||^{\mathfrak{H}}_{H^{\frac{1}{2}}(\partial U)}$ and $||\cdot||^{G,\mathfrak{H}}_{H^{\frac{1}{2}}(\partial U)}$ on $H^{\frac{1}{2}}(\partial U, d\sigma)$, the homomorphism from $\mathscr{H}(U)$ to $\mathscr{H}(G)$, given by (10.6), is in fact an isomorphism and its inverse is described such as

$$\mathcal{E}^{\mathfrak{H}}_{\mathfrak{e}} \circ \gamma : \mathscr{H}(G) \to \mathscr{H}(U). \tag{10.11}$$

In addition, $H^{-\frac{1}{2}}(\partial U, d\sigma)$ is also a real Hilbert function space with respect to

$$\langle g, h\rangle^{\mathfrak{H}}_{H^{-\frac{1}{2}}(\partial U)} := \sum_{k=1}^{\infty} \left(\frac{1}{\sqrt{1+\delta_k}} \int_{\partial U} g\gamma s_k \, d\sigma\right) \left(\frac{1}{\sqrt{1+\delta_k}} \int_{\partial U} h\gamma s_k \, d\sigma\right), \quad (10.12)$$

and the associated norm is denoted $||g||^{\mathfrak{H}}_{H^{-\frac{1}{2}}(\partial U)}$, which is equivalent to $||g||^{G,\mathfrak{H}}_{H^{-\frac{1}{2}}(\partial U)}$. For more details on the spaces $H^s(\partial U, d\sigma)$ $(s \in \mathbb{R})$, see [4, 5, 12].

On the other hand, using the interior and exterior regularized harmonic Steklov eigenvalues $\left\{\tau^G_j\right\}_{j=0}^{\infty}$ and $\{\tau_k\}_{k=1}^{\infty}$, together with associated eigenfunctions $\left\{u^G_j\right\}_{j=0}^{\infty}$ (see (9.25)) and $\{u_k\}_{k=1}^{\infty}$, $H^{\frac{1}{2}}(\partial U, d\sigma)$, with respect to the inner products

$$\begin{cases} \langle g, h\rangle^{G,\mathfrak{R}}_{H^{\frac{1}{2}}(\partial U)} := \sum_{j=0}^{\infty} \left(\sqrt{\tau^G_j} \int_{\partial U} g\gamma u^G_j \, d\sigma\right) \left(\sqrt{\tau^G_j} \int_{\partial U} h\gamma u^G_j \, d\sigma\right) \\ \text{and} \\ \langle g, h\rangle^{\mathfrak{R}}_{H^{\frac{1}{2}}(\partial U)} := \sum_{k=1}^{\infty} \left(\sqrt{\tau_k} \int_{\partial U} g\gamma u_k \, d\sigma\right) \left(\sqrt{\tau_k} \int_{\partial U} h\gamma u_k \, d\sigma\right), \end{cases} \quad (10.13)$$

is a real Hilbert function space, with the generated norms $||g||^{G,\mathfrak{R}}_{H^{\frac{1}{2}}(\partial U)}$ and $||g||^{\mathfrak{R}}_{H^{\frac{1}{2}}(\partial U)}$ which are equivalent on $H^{\frac{1}{2}}(\partial U, d\sigma)$ through analogous discussions.

For all $g \in H^{\frac{1}{2}}(\partial U, d\sigma)$, let

$$\begin{cases} \tilde{\mathfrak{k}}_g := \sum_{j=0}^{\infty} \sqrt{\tau^G_j} \left\langle g, \gamma u^G_j\right\rangle_{2,\partial U} \cdot \mathfrak{u}^G_j \\ \text{and} \\ \mathfrak{k}_g := \sum_{k=1}^{\infty} \sqrt{\tau_k} \left\langle g, \gamma u_k\right\rangle_{2,\partial U} \cdot \mathfrak{u}_k \end{cases} \quad (10.14)$$

be the respective unique *interior* and *exterior regularized harmonic extensions* of g over G and U. Then, we have $\left|\left|\tilde{\mathfrak{k}}_g\right|\right|_{H^1(G)} = ||g||^{G,\mathfrak{R}}_{H^{\frac{1}{2}}(\partial U)}$ and $||\mathfrak{k}_g||_{H^1(U)} = ||g||^{\mathfrak{R}}_{H^{\frac{1}{2}}(\partial U)}$.

Accordingly, we can obtain the following isometric isomorphisms

$$\begin{cases} \gamma : \mathscr{N}_1(G) \to H^{\frac{1}{2}}(\partial U, d\sigma) \text{ and } \mathcal{E}_{\mathfrak{i}}^{\mathfrak{R}} : H^{\frac{1}{2}}(\partial U, d\sigma) \to \mathscr{N}_1(G), \\ \\ \gamma : \mathscr{N}_1(U) \to H^{\frac{1}{2}}(\partial U, d\sigma) \text{ and } \mathcal{E}_{\mathfrak{e}}^{\mathfrak{R}} : H^{\frac{1}{2}}(\partial U, d\sigma) \to \mathscr{N}_1(U), \end{cases} \tag{10.15}$$

with $\tilde{\mathfrak{k}}_g = \mathcal{E}_{\mathfrak{i}}^{\mathfrak{R}}(g)$ and $\mathfrak{k}_g = \mathcal{E}_{\mathfrak{e}}^{\mathfrak{R}}(g)$, respectively, and the isomorphisms below

$$\mathcal{E}_{\mathfrak{i}}^{\mathfrak{R}} \circ \gamma : \mathscr{N}_1(U) \to \mathscr{N}_1(G) \text{ and } \mathcal{E}_{\mathfrak{e}}^{\mathfrak{R}} \circ \gamma : \mathscr{N}_1(G) \to \mathscr{N}_1(U). \tag{10.16}$$

Moreover, $H^{-\frac{1}{2}}(\partial U, d\sigma)$ is a real Hilbert function space with respect to

$$\begin{cases} \langle g, h\rangle^{G,\mathfrak{R}}_{H^{-\frac{1}{2}}(\partial U)} := \sum\limits_{j=0}^{\infty} \left(\frac{1}{\sqrt{\tau_j^G}} \int_{\partial U} g\gamma u_j^G \, d\sigma \right) \left(\frac{1}{\sqrt{\tau_j^G}} \int_{\partial U} h\gamma u_j^G \, d\sigma \right) \\ \text{or} \\ \langle g, h\rangle^{\mathfrak{R}}_{H^{-\frac{1}{2}}(\partial U)} := \sum\limits_{k=1}^{\infty} \left(\frac{1}{\sqrt{\tau_k}} \int_{\partial U} g\gamma u_k \, d\sigma \right) \left(\frac{1}{\sqrt{\tau_k}} \int_{\partial U} h\gamma u_k \, d\sigma \right), \end{cases} \tag{10.17}$$

with the equivalent norms denoted $||g||^{G,\mathfrak{R}}_{H^{-\frac{1}{2}}(\partial U)}$ and $||g||^{\mathfrak{R}}_{H^{-\frac{1}{2}}(\partial U)}$, respectively.

Finally, via (3.25), (7.29) and the fact $||\cdot||_{H^1(G)} \sim ||\cdot||_{\partial,G}$ on $H^1(G)$, one has

$$\mathscr{H}(G) \cong H^1(G) \big/ H_0^1(G) \cong \mathscr{N}_1(G). \tag{10.18}$$

This further implies that all these spaces $\mathscr{H}(G)$, $\mathscr{N}_1(G)$, $\mathscr{H}(U)$, and $\mathscr{N}_1(U)$ in fact are isomorphic to one another using composition of the trace mapping γ with its suitable inverses $\mathcal{E}_{\mathfrak{i}}^{\mathfrak{H}}$, $\mathcal{E}_{\mathfrak{i}}^{\mathfrak{R}}$, $\mathcal{E}_{\mathfrak{e}}^{\mathfrak{H}}$, and $\mathcal{E}_{\mathfrak{e}}^{\mathfrak{R}}$, respectively, and that all these norms on $H^{\frac{1}{2}}(\partial U, d\sigma)$ and all those on $H^{-\frac{1}{2}}(\partial U, d\sigma)$ are equivalent, respectively.

Appendix

We now show below how exactly (3.19) and (3.21) derives that

$$\int_{\partial U} \left(\gamma u_{k_1} - \gamma u_{k_2}\right)\left(D_\nu \tilde{u}_{k_1} - D_\nu \tilde{u}_{k_2}\right) d\sigma \to 0 \tag{a.1}$$

when $k_1, k_2 \to \infty$.

Actually, via the interior harmonic Steklov eigenvalues $\left\{\delta_j^G\right\}_{j=0}^{\infty}$ and associated eigenfunctions $\left\{s_j^G\right\}_{j=0}^{\infty}$ over G, any sequence $\{g_k\}_{k=1}^{\infty}$ of functions in $H^{\frac{1}{2}}\left(\partial U, d\sigma\right)$, decaying to zero as $k \to \infty$ in the norm $||\cdot||_{2,\partial U}$, can be rephrased as

$$g_k = \sum_{j=0}^{\infty} \left\langle g_k, \gamma s_j^G \right\rangle_{2,\partial U} \cdot \gamma s_j^G \tag{a.2}$$

such that

$$||g_k||_{2,\partial U} = \sqrt{\sum_{j=0}^{\infty} \left|\left\langle g_k, \gamma s_j^G \right\rangle_{2,\partial U}\right|^2} \to 0 \tag{a.3}$$

as $k \to \infty$, and such that, for all $k = 1, 2, \ldots$,

$$||g_k||_{H^{\frac{1}{2}}(\partial U)}^{G,\mathfrak{H}} = \sqrt{\sum_{j=0}^{\infty} \left(1 + \delta_j^G\right) \left|\left\langle g_k, \gamma s_j^G \right\rangle_{2,\partial U}\right|^2} < \infty. \tag{a.4}$$

Moreover, noticing (9.2) and (10.2), for each $k = 1, 2, \ldots$, let

$$\tilde{v}_k := \sum_{j=0}^{\infty} \sqrt{1 + \delta_j^G} \left\langle g_k, \gamma s_j^G \right\rangle_{2,\partial U} \cdot \mathfrak{s}_j^G \tag{a.5}$$

be the unique interior harmonic extension of g_k over G.

We then have $||\tilde{v}_k||_{\partial,G} = ||g_k||^{G,\mathfrak{H}}_{H^{\frac{1}{2}}(\partial U)} < \infty$ by (10.1) and $(a.4)$.

In addition, on ∂U, one sees that

$$D_\nu \tilde{v}_k = \sum_{j=0}^{\infty} \left\langle g_k, \gamma s_j^G \right\rangle_{2,\partial U} \delta_j^G \cdot \gamma s_j^G, \tag{$a.6$}$$

which together with (10.5) further implies that

$$||D_\nu \tilde{v}_k||^{G,\mathfrak{H}}_{H^{-\frac{1}{2}}(\partial U)} = \sqrt{\sum_{j=0}^{\infty} \frac{\left| \left\langle g_k, \gamma s_j^G \right\rangle_{2,\partial U} \delta_j^G \right|^2}{1+\delta_j^G}} \leq ||g_k||^{G,\mathfrak{H}}_{H^{\frac{1}{2}}(\partial U)} < \infty. \tag{$a.7$}$$

Now, resorting to a subsequence if necessary, it follows that

$$\int_{\partial U} (\gamma \tilde{v}_k)(D_\nu \tilde{v}_k)\, d\sigma = \sum_{j=0}^{\infty} \delta_j^\Omega \left| \left\langle g_k, \gamma s_j^G \right\rangle_{2,\partial U} \right|^2 \to 0 \tag{$a.8$}$$

when $k \to \infty$. This can be proved in exactly the same manner as those discussions shown in between (7.8) and (7.10), since now we have $\left(1+\delta_j^G\right) \left| \left\langle g_k, \gamma s_j^G \right\rangle_{2,\partial U} \right|^2 \to 0$ via $(a.3)$ and $(a.4)$ when $k \to \infty$, for any fixed $j = 0, 1, 2, \ldots$.

Note here, along with certain arguments in the preceding chapter, we actually have proved that all sequences $\{g_k\}_{k=1}^{\infty}$ of functions in the space $H^{\frac{1}{2}}(\partial U, d\sigma)$ are Cauchy with respect to $||\cdot||^{G,\mathfrak{H}}_{H^{\frac{1}{2}}(\partial U)}$, if and only if they are Cauchy with respect to $||\cdot||^{\mathfrak{H}}_{H^{\frac{1}{2}}(\partial U)}$, and if and only if they are Cauchy with respect to $||\cdot||_{2,\partial U}$. Nevertheless, the completeness of the space $H^{\frac{1}{2}}(\partial U, d\sigma)$ is derived either from that of $H^1(G)$ or from that of $E^1(U)$, but clearly not from that of $L^2(\partial U, d\sigma)$.

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