# STABLE PHASE RETRIEVAL USING LOW-REDUNDANCY FRAMES OF POLYNOMIALS

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A Dissertation

Presented to

the Faculty of the Department of Mathematics

University of Houston

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In Partial Fulfillment

of the Requirements for the Degree

Doctor of Philosophy

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By

Nathaniel Hammen

December 2015

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# Abstract

In many applications, measurements of a signal consist of the magnitudes of linear functionals while the phase information of these functionals is unavailable. Examples of these type of measurements occur in optics, quantum mechanics, speech recognition, and x-ray crystallography. The main topic of this dissertation is the recovery of the phase information of a signal using a small number of these magnitude measurements. This is called phase retrieval. We provide a choice of $4d-4$ magnitude measurements that uniquely determines any $d$ dimensional signal, up to a unimodular constant. Then we provide a choice of $6d-3$ magnitude measurements that admits a stable polynomial time algorithm to recover the signal under the influence of noise. We also explore the behavior of pathological signals in this algorithm, as well as the mean squared error. Finally, we show that if the signal is known to be $s$ sparse, then we only need a suitable choice of $O(s\log d/s)$ such measurements for the stable algorithm to successfully recover the signal.

# Contents

# List of Figures

# Chapter 1

# Introduction

## 1.1 Motivation

In the topic of signal recovery, signals are often treated as vectors in a Hilbert space, and measurements of a signal are treated as inner products with some set of measurement vectors that does not depend on the signal vector. In the simplest examples, recovery of a signal from its measurements is equivalent to solving the familiar linear algebra equation $Ax = b$ for the vector $x$. However, in many applications an intensity may be measured while any phase information is unavailable. In this case, phaseless measurements of the signal would be represented by the magnitudes of real or complex inner products with some set of measurement vectors. The problem of recovering a signal from such measurements is called phase retrieval. The phase retrieval problem for real-valued signals has been well studied [6, 7]. Because coefficients have only two possible phases, this problem has a combinatorial character. Here, we study the phase retrieval problem for complex Hilbert spaces. The phase retrieval problem is equivalent to solving the equation $|Ax|^2 = b$ for the vector $x$, where the square of the complex modulus is taken separately on each component.

When solving the phase retrieval problem, if $c \in \mathbb{C}$ such that $|c| = 1$, then $x$ and $cx$ are indistinguishable. Thus, solving for $x$ up to multiplication by a unimodular constant is the most precise solution that can be achieved. Even when attempting to solve for such equivalence classes, the problem still presents challenges. Due to the nonlinearity of the phase retrieval problem, there may be choices of measurements for which the map that takes the signal to the phaseless measurements is injective, but a feasible algorithm for recovery of the signal from these measurements cannot be found. And even if such an algorithm does exist, it may not be stable. A larger number of measurements is required for the measurement map to be injective, when compared to the linear case, because of the loss of phase information. We are interested in finding a small set of measurement vectors such that the map from the signal vector to the complex modulus of the inner products with the measurement vectors is injective, permits a feasible algorithm that recovers the signal from its measurements, and the algorithm is stable with respect to noise in the measurements.

There are multiple applications for which a solution to the phase retrieval problem would prove useful, most of which are related to the Fourier transform [1, 2, 36]. In X-ray crystallography, measurements are taken of the intensity of the Fourier transform of an object [17, 55, 57]. A similar situation also arises in optics, for example when imaging stars through a telescope [67]. In quantum mechanics, the probability of observing a particular outcome can be represented as the square of the complex modulus of an inner product of the state vector with a measurement vector [37, 38, 44, 46]. A common procedure for speech recognition and enhancement measures the short time Fourier transform and then modifies the magnitudes of the Fourier coefficients, while any corresponding modifications to the phase are undetermined [7, 61, 62]. In all of these applications, attempts at recovery with flawed phase information causes distortions to the recovered signal.

## 1.2 Background and summary

An important problem in phase retrieval is the problem of finding the smallest possible set of measurement vectors such that the map that takes the signal to the phaseless measurements is injective. Similarly, we would also like to find a small set of measurement vectors such that a stable algorithm exists to recover the signal from the phaseless measurements. It has been known for over a decade that, in the case in which the $d$ dimensional signal has no zero coefficients with respect to a chosen basis, there exists a set of $3d-2$ measurements which suffices to recover the signal and that this number is minimal [37, 38]. However, the assumption that the signal has no zero components is too limiting to be acceptable in standard signal models.

Until this past year, it was conjectured that $4d-4$ measurements would permit injectivity of the measurement map over all signals, and that this is the minimum such number of measurements [9]. In a result from Bernhard Bodmann and myself, a set of $4d-4$ measurement vectors was constructed such that the measurement map is injective [14]. This shows that $4d-4$ measurements can permit injectivity of the measurement map, but this does not imply that $4d-4$ is a minimal number of measurements that can permit injectivity. In fact, a counterexample was recently found by Cynthia Vinzant, who showed that for $d = 4$ there exists a system of 11 measurements that is injective [64]. The minimal number of measurements required is not known, but a lower bound on the minimum number of measurements of $4d - O(\log(d))$ has been shown [44].

Recent techniques for solving the phase retrieval problem have focused on what are called lifting procedures. Instead of trying to solve the phase retrieval problem in the space of vectors, where this problem is nonlinear, we injectively map to a higher dimensional space in which the phaseless measurements are linear. A popular choice of lifting procedure is the mapping $x \mapsto xx^*$ [4, 5, 7, 19–21, 25]. This choice works because for any two vectors $x$

and $y$, $|\langle x, y\rangle|^2 = \mathrm{tr}(xx^*yy^*)$. Thus, measurements that were the square of the complex modulus of an inner product are equal to inner products of rank-one Hermitian matrices, and so are linear measurements in the space of matrices. However, if $x$ is $d$ dimensional, then the space of matrices is $d^2$ dimensional, and we would like to find a set of phaseless measurements that is linear in number. So this space is too large to uniquely determine any matrix using a linear number of measurements. However, only rank-one Hermitian matrices need to be considered, and although rank minimization is known to be NP-complete [63], additional knowledge about the signal or additional requirements on the measurements can allow rank-one Hermitian matrices to be uniquely determined using a linear number of measurements.

In some applications in quantum mechanics, the problem does not exactly match the phase retrieval problem being explored here, because the measurements are not required to be rank-one. Instead, the measurements are allowed to be what are called positive operator valued measures [26, 44]. In this setting, any signal may be reconstructed using $4d - O(\log(d))$ of these more general forms of measurements [44] and this number is minimal. In fact, this implies that the rank-one measurements cannot reconstruct a signal with fewer measurements than in the POVM setting, and so gives a lower bound for the minimal number of measurements required to solve the phase retrieval problem.

One method that allows the space of matrices to be used to solve the phase retrieval problem is to demand a stronger requirement on the set of measurements than injectivity over the rank-one Hermitian matrices. For example, the PhaseLift algorithm requires a set of measurements such that for any rank-one Hermitian matrix $X$ and any positive semi-definite $Y$, if the measurements on $X$ and $Y$ are equal, then $X = Y$. In this setting semi-definite programming may be used to recover $xx^*$ from these measurements [19–21, 25, 41]. However, most examples of measurements that satisfy this requirement are either not linear in number, or are linear in number with a large constant. One exception to this is a choice of $5d-6$ measurements satisfying this requirement that has been found in the past few months [50]. At the moment, this is the smallest known set of measurements that admits a stable phase retrieval algorithm.

In this dissertation we are interested in a different choice of lifting space. If the signal is represented as a polynomial $p$, then the trigonometric polynomial that is given by $\overline{p(z)}p(z)$ has linear measurements that are equal to the phaseless measurements on $p$. However, this mapping is not injective. Instead we look at mapping the polynomial $p$ to both $\overline{p(z)}p(z)$ and $\overline{Tp(z)}Tp(z)$ for some linear transformation $T$. This can be shown to be injective for an appropriate choice of $T$. In section 2.2, we provide a proof that shows that if we know the magnitudes of the point evaluation measurements of $p$ on two circles that intersect at an irrational angle, then we have an injective map into a $4d-4$ dimensional space, and so have an injective set of $4d-4$ measurements; this work appeared in [14] with Bodmann. Another example of the use of polynomial spaces to obtain an injective set of measurements comes from Friedrich Philipp [58]. His procedure required measuring $p$ at $2d-1$ equally space points on the unit circle, and measuring the derivative $p'$ at $2d-3$ equally spaced points on the unit circle. However, neither of these methods admits a phase retrieval algorithm with recovery error that is linear in terms of noise. In section 2.3, we provide a phase retrieval algorithm in $6d-3$ measurements using polynomial spaces, and prove that it is stable in section 2.6; this work appeared in [15] with Bodmann.

Often, there are fewer measurements that are feasibly available than the dimension of the signal to be recovered. The problem of recovering a sparse signal from fewer linear measurements than the dimension of the signal is called compressive sensing. It would be useful to combine phase retrieval results with compressive sensing results. This idea of combining these two problems has been explored in recent years [11, 13, 31, 42, 45, 52, 65].

Some of these methods have provable performance guarantees in the presence of noise, but they do not include precise conditions on the number of measured quantities that are sufficient [45, 52, 65]. We provide an example of compressive phase retrieval that does have both provable performance guarantees and precise conditions on the number of measured quantities that are sufficient in section 3.3; this work appeared in [16] with Bodmann.

## 1.3 Preliminaries

We begin by establishing notation for sets that will be frequently used.

**Definition 1.3.1.** The set of unimodular complex numbers $\{z \in \mathbb{C} : |z| = 1\}$ is denoted as $\mathbb{T}$.

In solving the phase retrieval problem, we are only interested in solving for $x$ up to multiplication by $\mathbb{T}$, as solving for $x$ is impossible.

**Definition 1.3.2.** The equivalence class of $x \in \mathbb{C}^d$ up to multiplication by a unimodular constant is denoted as $[x]$. The set of all such equivalence classes is denoted as $\mathbb{C}^d/\mathbb{T}$.

### 1.3.1 Polynomial spaces

Various polynomial spaces will be used frequently in our results, so we define and establish notation for these spaces.

**Definition 1.3.3.** The space of analytic polynomials of degree less than $d$ is defined as the space of functions on $\mathbb{C}$ that can be represented by the map $z \mapsto \sum_{j=0}^{d-1} x_j z^j$, for some vector $(x_j)_{j=0}^{d-1} \in \mathbb{C}^d$. This space is denoted as $\mathcal{P}_d$. It is equipped with the inner product induced by the scaled Lebesgue measure on $\mathbb{T}$, so $p, q \in \mathcal{P}_d$ have the inner product

$$\langle p, q \rangle = \frac{1}{2\pi} \int_{[0,2\pi]} p(e^{it})\overline{q(e^{it})}dt\,.$$

Consequently, the polynomials $\{z \mapsto z^j\}_{j=0}^{d-1}$ form the standard orthonormal basis for $\mathcal{P}_d$. Thus, there exists a natural isometric isomorphism between $\mathbb{C}^d$ and $\mathcal{P}_d$ via the map that takes $x \in \mathbb{C}^d$ to the polynomial represented by the map $z \mapsto \sum_{j=0}^{d-1} x_j z^j$. For any $x \in \mathbb{C}^d$, we denote this resulting polynomial as $p_x$, and note that for any polynomial $p \in \mathcal{P}_d$ there exists a unique $x \in \mathbb{C}^d$ such that $p = p_x$. Note that by this isometry, we have that $\langle x, y\rangle = \langle p_x, p_y\rangle$ for any $x, y \in \mathbb{C}^d$.

A useful property of the space of analytic polynomials is the reproducing property, which is the fact that on this space any point evaluation is a linear functional. To establish this property we shall define a set of kernel functions, and show that any point evaluation is equal to the inner product with one of these kernel functions.

**Definition 1.3.4.** For any $w \in \mathbb{C}$ we may define the kernel polynomial $K_w \in \mathcal{P}_d$ as the polynomial such that $K_w(z) = \sum_{j=0}^{d-1} \overline{w^j} z^j$.

In particular, $K_0$ is the polynomial such that $K_0(z) = 1$ for all $z \in \mathbb{C}$. Using these kernel polynomials, we shall show that the point evaluation of $p \in \mathcal{P}_d$ at $w \in \mathbb{C}$ is equal to the inner product with $K_w$.

**Proposition 1.3.5.** *For any $p \in \mathcal{P}_d$, if $x \in \mathbb{C}^d$ such that $p = p_x$, then*

$$p(w) = \sum_{j=0}^{d-1} x_j w^j = \langle p, K_w\rangle \,.$$

An additional useful property is the fact that there exists a basis composed entirely of these kernel functions.

**Proposition 1.3.6.** *If $\omega_d = e^{\frac{2i\pi}{d}}$ is the $d$-th root of unity, then for any $z_0 \in \mathbb{T}$, the polynomials $\{\frac{1}{\sqrt{d}} K_{z_0\omega_d^j}\}_{j=0}^{d-1}$ form an orthonormal basis for $\mathcal{P}_d$. Thus, any polynomial $p \in \mathcal{P}_d$ satisfies $p = \sum_{j=0}^{d-1} p(z_0\omega_d^j) \frac{1}{d} K_{z_0\omega_d^j}$.*

We will be representing the signal to be recovered as an element of $\mathcal{P}_d$. But as previously stated, the phase retrieval problem can only be solved up to multiplication by a unimodular constant.

**Definition 1.3.7.** The equivalence class of $p \in \mathcal{P}_d$ up to multiplication by a unimodular constant is denoted as $[p]$. The set of all such equivalence classes is denoted as $\mathcal{P}_d/\mathbb{T}$.

Note that for any linear operator $T : \mathcal{P}_d \to \mathcal{P}_d$, the map $[p] \mapsto [Tp]$ is well defined, and if $T$ is a bijection, then this new map is also a bijection. One useful transformation on polynomials that will show up frequently is the map that composes the input with a chosen linear polynomial.

**Definition 1.3.8.** For any $c, r \in \mathbb{C}$ and polynomial $p \in \mathcal{P}_d$, the map $z \mapsto p(c+rz)$ is itself a polynomial in $\mathcal{P}_d$. We define $R_{c,r}$ to be the function that sends any polynomial $p$ to the polynomial represented by $z \mapsto p(c+rz)$.

The function $R_{c,r}$ is a linear operator for any choice of $c$ and $r$ in $\mathbb{C}$, and it is a bijective linear operator if and only if $r \neq 0$. In fact, $R_{0,1}$ is equal to the identity operator.

Next we consider the space of trigonometric polynomials.

**Definition 1.3.9.** The space of trigonometric polynomials of degree less than $d$ is defined as the space of functions on $\mathbb{T}$ that can be represented by the map $z \mapsto \sum_{j=-(d-1)}^{d-1} y_j z^j$, for some vector $(y_j)_{j=-(d-1)}^{d-1} \in \mathbb{C}^{2d-1}$. This space is denoted as $\mathcal{T}_d$. This space is also equipped with the inner product induced by the scaled Lebesgue measure on $\mathbb{T}$, so $f, g \in \mathcal{T}_d$ have the inner product

$$\langle f, g \rangle = \frac{1}{2\pi} \int_{[0,2\pi]} f(e^{it})\overline{g(e^{it})} dt \,.$$

Consequently, the polynomials $\{z \mapsto z^j\}_{j=-(d-1)}^{d-1}$ form the standard orthonormal basis for $\mathcal{T}_d$.

Just as with the analytic polynomials, a useful property of the space of trigonometric polynomials is the reproducing property, which is the fact that on this space any point evaluation is a linear functional. To establish this property we shall define a set of kernel functions, and show that any point evaluation is equal to the inner product with one of these kernel functions.

**Definition 1.3.10.** For any $w \in \mathbb{T}$ we may define the $w$-rotated Dirichlet kernel $D_{w,d-1} \in \mathcal{T}_d$ as the trigonometric polynomial such that $D_{w,d-1}(z) = \sum_{j=-(d-1)}^{d-1} \overline{w^j} z^j$.

Using these rotated Dirichlet kernels, we shall show that the point evaluation of $f \in \mathcal{T}_d$ at $w \in \mathbb{T}$ is equal to the inner product with $D_{w,d-1}$.

**Proposition 1.3.11.** *For any $f \in \mathcal{T}_d$, if $f$ is represented by the map $z \mapsto \sum_{j=-(d-1)}^{d-1} y_j z^j$, then*

$$f(w) = \sum_{j=-(d-1)}^{d-1} c_j w^j = \langle f, D_{w,d-1} \rangle .$$

An additional useful property is the fact that there exists a basis composed entirely of these kernel functions.

**Proposition 1.3.12.** *If $\omega_{2d-1} = e^{\frac{2i\pi}{2d-1}}$ is the $2d-1$-st root of unity, then the polynomials $\{\frac{1}{\sqrt{2d-1}} D_{\omega_{2d-1}^j, d-1}\}_{j=0}^{2d-2}$ form an orthonormal basis for $\mathcal{T}_d$, which we call the Dirichlet kernel basis. Thus, any polynomial $f \in \mathcal{T}_d$ satisfies $f = \sum_{j=0}^{2d-2} f(\omega_{2d-1}^j) \frac{1}{2d-1} D_{\omega_{2d-1}^j, d-1}$. This is called Dirichlet kernel interpolation.*

We also define a subset of $\mathcal{T}_d$ that is analogous to the set of Hermitian matrices.

**Definition 1.3.13.** The set of real trigonometric polynomials of degree less than $d$ is defined as the set of functions on $\mathbb{T}$ that can be represented by the map $z \mapsto \sum_{j=-(d-1)}^{d-1} y_j z^j$, for some vector $(y_j)_{j=-(d-1)}^{d-1} \in \mathbb{C}^{2d-1}$ with the additional property that $y_{-j} = \overline{y}_j$ for all $j$. This set is denoted as $\mathcal{R}_d$ and is a subset of $\mathcal{T}_d$.

Note that $\mathcal{R}_d$ is a vector space over $\mathbb{R}$ but not over $\mathbb{C}$. In fact, for any $f, g \in \mathcal{R}_d$, decomposition into the standard orthonormal basis shows that $\langle f, g\rangle \in \mathbb{R}$.

**Proposition 1.3.14.** *For $f \in \mathcal{T}_d$, we have that $f \in \mathcal{R}_d$ if and only if $f(z) \in \mathbb{R}$ for all $z \in \mathbb{T}$.*

*Proof.* First, note that for any $w \in \mathbb{T}$, the definition of the Dirichlet kernel gives that $D_{w,d-1} \in \mathcal{R}_d$, so that every element of the Dirichlet kernel basis is in $\mathcal{R}_d$. Thus if $f(z) \in \mathbb{R}$ for all $z \in \mathbb{T}$, then Dirichlet kernel interpolation gives $f = \sum_{j=0}^{2d-2} f(\omega_{2d-1}^j)\frac{1}{2d-1}D_{\omega_{2d-1}^j,d-1}$, which is a real linear combination of elements in $\mathcal{R}_d$, so $f \in \mathcal{R}_d$. On the other hand, if $f \in \mathcal{R}_d$, then for any $z \in \mathbb{T}$, $f(z) = \langle f, D_{w,d-1}\rangle \in \mathbb{R}$. $\square$

# Chapter 2

# Phase retrieval

## 2.1 Comparison of lifting spaces

A useful way to represent the phase retrieval problem is to find a set of vectors $\{\nu_j\}$ in $\mathbb{C}^d$ such that any vector $x \in \mathbb{C}^d$ can be recovered from the measurements $|\langle x, \nu_j \rangle|^2$. We define phaseless measurements in this fashion.

**Definition 2.1.1.** We call a function $b : \mathbb{C}^d \to \mathbb{R}$ a phaseless measurement if there exists $\nu \in \mathbb{C}^d$ such that $b(x) = |\langle x, \nu \rangle|^2$ for all $x \in \mathbb{C}^d$. Given a set $\{b_j\}_{j=0}^{M-1}$ of phaseless measurements, the vectors that correspond to these functions are called the measurement vectors.

This representation will allow us to injectively map into a higher dimensional linear space in which these measurements are equal to linear measurements. We call this a lifting procedure. Before this is demonstrated, we need to define the complex conjugate of a polynomial.

**Definition 2.1.2.** Also note that if $p$ is represented by the map $z \mapsto \sum_{j=0}^{d-1} x_j z^j$, then the map $z \mapsto \sum_{j=0}^{d-1} \overline{x}_j z^j$ is itself a polynomial. We denote this new polynomial by $\overline{p}$.

With the conjugate defined, we may now define the trigonometric polynomial lifting.

**Definition 2.1.3.** For any $p \in \mathcal{P}_d$, we define $|p|^2 \in \mathcal{T}_d$ to be the trigonometric polynomial represented by the map $z \mapsto \overline{p}(z^{-1})p(z)$. We call $|p|^2$ the trigonometric polynomial lifting of $p$. Note that for any $z_0 \in \mathbb{T}$, we have $|z_0 p|^2 = |p|^2$, so the map $[p] \mapsto |p|^2$ is well defined.

The trigonometric polynomial lifting of a polynomial $p \in \mathcal{P}_d$ is useful because its values on $\mathbb{T}$ are equal to the square of the complex modulus of the values of $p$ at the same points. Because point evaluation is an inner product with a vector as in proposition 1.3.5, the values of the trigonometric polynomial lifting are equal to phaseless measurements.

**Proposition 2.1.4.** *For any polynomial $p \in \mathcal{P}_d$, its trigonometric polynomial lifting $|p|^2$ is in $\mathcal{R}_d$, and for any $z \in \mathbb{T}$, $|p|^2(z) = |p(z)|^2$.*

*Proof.* Note that for any $z \in \mathbb{T}$, we have $\overline{z} = z^{-1}$, so

$$|p|^2(z) = \overline{p}(z^{-1})p(z) = \overline{p}(\overline{z})p(z) = \overline{p(z)}p(z) = |p(z)|^2 .$$

This also shows that $|p|^2$ only takes real values on $\mathbb{T}$, and so $|p|^2 \in \mathcal{R}_d$. $\square$

In fact, trigonometric polynomial liftings allow us to obtain a trigonometric polynomial that has values on $\mathbb{T}$ that are equal to the values of any chosen analytic polynomial on any chosen circle.

**Corollary 2.1.5.** *For any $c, r \in \mathbb{C}$ and any $p \in \mathcal{P}_d$, the trigonometric polynomial lifting $|R_{c,r}p|^2$ satisfies $|R_{c,r}p|^2(z) = |R_{c,r}p(z)|^2 = |p(c + rz)|^2$.*

Another choice of lifting procedure that is often used is the idea of lifting $[x] \in \mathbb{C}^d/\mathbb{T}$ to $xx^*$ in the space of matrices. For both choices of lifting procedure, any phaseless measurements of $x$ are equal to linear measurements in the lifting space.

**Proposition 2.1.6.** *Given a vector $x \in \mathbb{C}^d$, and a set of functions $b_j : \mathbb{C}^d \to \mathbb{R}$ for $j$ from $0$ to $M-1$, the following are equivalent:*

1. *For $j$ from $0$ to $M-1$, there exist vectors $\nu_j \in \mathbb{C}^d$ such that $|\langle x, \nu_j \rangle|^2 = b_j(x)$.*

2. *For $j$ from $0$ to $M-1$, there exist Hermitian matrices $Y_j \in \mathbb{C}^{d\times d}$ with rank at most one such that $\operatorname{tr}(Y_j x x^*) = b_j(x)$.*

3. *For $j$ from $0$ to $M-1$, there exist linear operators $T_j : \mathcal{P}_d \to \mathcal{P}_d$ and unimodular $z_j \in \mathbb{C}$ such that $|T_j p_x|^2(z_j) = b_j(x)$.*

*Proof.* $1 \iff 2$ Note that for any $j$, a Hermitian matrix $Y_j \in \mathbb{C}^{d\times d}$ has rank at most one if and only if there exists a vector $f_j \in \mathbb{C}^d$ such that $Y_j = f_j f_j^*$. In this case

$$|\langle x, f_j \rangle|^2 = f_j^* x x^* f_j = \operatorname{tr}(f_j^* x x^* f_j) = \operatorname{tr}(f_j f_j^* x x^*) = \operatorname{tr}(Y_j x x^*)$$

and so $|\langle x, f_j \rangle|^2 = \operatorname{tr}(Y_j x x^*) = b_j(x)$.

$1 \implies 3$ If for each $j$, we let $T_j$ be the linear transformation that sends $p_x$ to $\langle x, f_j \rangle K_0$, then for any choice of $z_j \in \mathbb{C}$ with $|z_j| = 1$,

$$|T_j p_x|^2(z_j) = |T_j p_x(z_j)|^2 = |\langle x, f_j \rangle K_0(z_j)|^2 = |\langle x, f_j \rangle|^2 = b_j(x)\,.$$

$3 \implies 1$ By the isomorphism between $\mathbb{C}^d$ and $\mathcal{P}_d$, we know that for each $j$ there exists a unique $f_j \in \mathbb{C}^d$ such that $T_j^* K_{z_j} = p_{f_j}$. Thus,

$$|\langle x, f_j \rangle|^2 = |\langle p_x, T_j^* K_{z_j} \rangle|^2 = |\langle T_j p_x, K_{z_j} \rangle|^2 = |T_j p_x(z_j)|^2 = |T_j p_x|^2(z_j) = b_j(x)\,.$$

□

In the above proposition, both item 2 and item 3 are linear measurements on the corresponding spaces. Item 2 is a linear measurement because the definition of Hilbert-Schmidt inner product is precisely what is described. Item 3 is a linear measurement due to the properties of the Dirichlet kernel. Note that the space of matrices is $d^2$ dimensional, and we would like to find a set of phaseless measurements that is linear in number, for which any signal can be recovered. So this space is in some sense larger than we would prefer. It is possible to find a linear set of measurements that allows phase retrieval on this space, but we will take a different approach.

The space of trigonometric polynomials is $2d-1$ dimensional, so that is promising, but the map $[p] \mapsto |p|^2$ is not injective. For example, if $e_0$ and $e_1$ are elements of the standard basis for $\mathbb{C}^d$, then $|p_{e_0}|^2$ and $|p_{e_1}|^2$ are both equal to the constant trigonometric polynomial with constant value 1, but $[p_{e_0}] \neq [p_{e_1}]$. Thus, the space $\mathcal{P}_d$ is in some sense too small. However, it can be shown that for a some choices of linear operator $T : \mathcal{P}_d \to \mathcal{P}_d$, the map $[p] \mapsto (|p|^2, |Tp|^2)$ is injective. Because the $\omega_{2d-1}^j$-rotated Dirichlet kernel forms a basis for $\mathcal{T}_d$, and inner products with these Dirichlet kernels are precisely the measurements described in proposition 2.1.6, it is possible to recover $(|p|^2, |Tp|^2)$ using linear algebra. The space $\mathcal{T}_d \oplus \mathcal{T}_d$ is $4d-2$ dimensional, and so the number of measurements required to do this is linear in $d$. However, depending on the choice of $T$, it may be difficult to recover $[p]$ from $(|p|^2, |Tp|^2)$.

As an example of this, if $\omega_d = e^{\frac{2i\pi}{d}}$ is the $d$th root of unity, and $c \in \mathbb{C}$ is chosen such that $c$ is a real multiple of $\omega_d + 1$ and $\arg(c-1)$ is an irrational multiple of $\pi$, then the map $[p] \mapsto (|p|^2, |R_{c,|c-1|}p|^2)$ is injective. This choice of transformation also gives the additional constraints $|p|^2(1) = |R_{c,|c-1|}p|^2(-\frac{c-1}{|c-1|})$ and $|p|^2(\omega_d) = |R_{c,|c-1|}p|^2(-\frac{c-\omega_d}{|c-\omega_d|})$, which reduces the dimension to $4d-4$. This is described in more detail in the next section. However, even though there exists an injective stable algorithm to obtain $(|p|^2, |R_{c,|c-1|}p|^2)$, an algorithm has not yet been discovered that can recover $[p]$ from $(|p|^2, |R_{c,|c-1|}p|^2)$ with an explicit error bound.

## 2.2 Injectivity in $4d-4$ measurements

For any polynomial $p \in \mathcal{P}_d$ and any linear operator $T : \mathcal{P}_d \to \mathcal{P}_d$, the coefficients of the two trigonometric polynomial liftings $|p|^2$ and $|Tp|^2$ with respect to the Dirichlet kernel basis are phaseless measurements, as in proposition 2.1.6. Thus, if it can be shown that the map $[p] \mapsto (|p|^2, |Tp|^2)$ is injective, then a number of phaseless measurements equal to the dimension of $(|p|^2, |Tp|^2)$ will suffice to give an injective set of measurements.

We shall show that for the appropriate choice of $c \in \mathbb{C}$, the map $[p] \mapsto (|p|^2, |R_{c,|c-1|}p|^2)$ is an injective map, and elements of the latter space can be recovered from phaseless measurements of $[p]$. We know that the values of $|R_{c,|c-1|}p|^2$ on $\mathbb{T}$ are equal to the modulus squared of the values of $p$ on the circle with center $c$ and radius $|c-1|$, as in corollary 2.1.5. Because the value used for the radius of the circle is $|c-1|$, this circle intersects $\mathbb{T}$ at 1. A Möbius transformation that maps 1 to the $\infty$ will be used to obtain values on two lines that correspond to $\mathbb{T}$ and the circle with center $c$ and radius $|c-1|$. Then the following result from Philippe Jaming [48] will provide injectivity.

**Theorem 2.2.1** (Polynomial case of Theorem 3.3 from [48])**.** *Let $\alpha_1, \alpha_2 \in [0, 2\pi)$ with $\alpha_1 - \alpha_2 \notin \pi\mathbb{Q}$ and $p, q \in \mathcal{P}_d$. If $|p(re^{i\alpha_1})| = |q(re^{i\alpha_1})|$ and $|p(re^{i\alpha_2})| = |q(re^{i\alpha_2})|$ for all $r \in \mathbb{R}$, then $[p] = [q]$.*

*Proof.* First, let us consider the case in which $p = 0$. Then $|q(re^{i\alpha_1})| = 0$ for all $r \in \mathbb{R}$, and so $q = 0$. Thus, $[p] = [q]$.

Otherwise, $p \neq 0$ and $q \neq 0$. Define $p_1, p_2, q_1, q_2 \in \mathcal{P}_{2d}$ such that for all $z \in \mathbb{C}$,

$$p_1(z) = p(ze^{i\alpha_1})\overline{p}(ze^{-i\alpha_1})\,,$$

$$p_2(z) = p(ze^{i\alpha_2})\overline{p}(ze^{-i\alpha_2})\,,$$

$$q_1(z) = q(ze^{i\alpha_1})\overline{q}(ze^{-i\alpha_1})\,,$$

and

$$q_2(z) = q(ze^{i\alpha_2})\overline{q}(ze^{-i\alpha_2})\,.$$

Then, for all $z \in \mathbb{R}$,

$$p_1(z) = p(ze^{i\alpha_1})\overline{p(ze^{i\alpha_1})} = |p(ze^{i\alpha_1})|^2 = |q(ze^{i\alpha_1})|^2 = q(ze^{i\alpha_1})\overline{q(ze^{i\alpha_1})} = q_1(z)$$

and similarly, $p_2(z) = q_2(z)$. Any polynomials that are equal on $\mathbb{R}$ are equal on $\mathbb{C}$, so $p_1 = q_1$ and $p_2 = q_2$. Note that $p_1(z) = 0$ if and only if either $p(ze^{i\alpha_1}) = 0$ or $\overline{p}(ze^{-i\alpha_1}) = 0$. But by complex conjugation, $\overline{p}(ze^{-i\alpha_1}) = 0$ if and only if $p(\overline{z}e^{i\alpha_1}) = 0$. Thus, $z$ is a root of $p_1$ if and only if either $ze^{i\alpha_1}$ or $\overline{z}e^{i\alpha_1}$ is a root of $p$. If $z$ is a root of $p_1$, then the multiplicity of the root is equal to the sum of the multiplicities of the roots of $p$ at $ze^{i\alpha_1}$ and $\overline{z}e^{i\alpha_1}$, where we consider multiplicity to be zero if there is no root. Similar statements hold for $p_2$, $q_1$, and $q_2$.

Let $\mathcal{Z}_p$ be the multiset of roots of $p$, and let $\mathcal{Z}_q$ be the multiset of roots of $q$. We shall show that $\mathcal{Z}_p = \mathcal{Z}_q$. By way of contradiction, assume that $\mathcal{Z}_p \neq \mathcal{Z}_q$. Then because $p_1 = q_1$, we know that $p$ and $q$ have the same number of roots, so $\mathcal{Z}_p \backslash \mathcal{Z}_q$ and $\mathcal{Z}_q \backslash \mathcal{Z}_p$ are both nonempty. Let $z_1 \in \mathcal{Z}_p \backslash \mathcal{Z}_q$. Then, $z_1 e^{-i\alpha_1}$ is a root of $p_1$ with multiplicity equal to the sum of the multiplicities of the roots of $p$ at $z_1$ and $\overline{z_1 e^{-i\alpha_1}} e^{i\alpha_1} = \overline{z}_1 e^{2i\alpha_1}$.

Because $p_1 = q_1$, $z_1 e^{-i\alpha_1}$ is a root of $q_1$ with the same multiplicity. Because $p$ has a higher multiplicity than $q$ for the root at $z_1$, $q$ must have a higher multiplicity than $p$ for the root at $\overline{z}_1 e^{2i\alpha_1}$. Thus, $\overline{z}_1 e^{2i\alpha_1} \in \mathcal{Z}_q \backslash \mathcal{Z}_p$. Then, $\overline{z}_1 e^{2i\alpha_1} e^{-i\alpha_2} = \overline{z}_1 e^{i(2\alpha_1 - \alpha_2)}$ is a root of $q_2$ with multiplicity equal to the sum of the multiplicities of the roots of $p$ at $\overline{z}_1 e^{2i\alpha_1}$ and $\overline{\overline{z}_1 e^{i(2\alpha_1-\alpha_2)}} e^{i\alpha_2} = z_1 e^{2i(\alpha_2 - \alpha_1)}$. Because $p_2 = q_2$, $\overline{z}_1 e^{i(2\alpha_1 - \alpha_2)}$ is a root of $p_2$ with the same multiplicity. Because $q$ has a higher multiplicity than $p$ for the root at $\overline{z}_1 e^{2i\alpha_1}$, $p$ must have a higher multiplicity than $q$ for the root at $z_1 e^{2i(\alpha_2-\alpha_1)}$. Thus, $z_1 e^{2i(\alpha_2 - \alpha_1)} \in \mathcal{Z}_p \backslash \mathcal{Z}_q$. Because this is true for any $z_1 \in \mathcal{Z}_p \backslash \mathcal{Z}_q$, we know that $\mathcal{Z}_p \backslash \mathcal{Z}_q$ is invariant under rotation by $2(\alpha_2 - \alpha_1)$, which is an irrational rotation. Thus, $\mathcal{Z}_p \backslash \mathcal{Z}_q$ has infinitely many points, and so $p$ is the zero polynomial. However, we are operating in the case that $p \neq 0$, so this is a contradiction. Thus $\mathcal{Z}_p = \mathcal{Z}_q$.

If $\mathcal{Z}_p = \mathcal{Z}_q$, then $p = wq$ for some $w \in \mathbb{C}$. Then $|p(re^{i\alpha_1})| = |q(re^{i\alpha_1})| = |wp(re^{i\alpha_1})|$ for all $r \in \mathbb{R}$, so $|w| = 1$. Thus, $[p] = [q]$. □

The above theorem shows that knowledge of the complex modulus of $p$ on two lines that intersect at 0 with an irrational angle is enough to determine $[p]$. In fact, we can allow the intersection point to lie anywhere in $\mathbb{C}$.

**Corollary 2.2.2.** *Let $\alpha_1, \alpha_2 \in [0, 2\pi)$ with $\alpha_1 - \alpha_2 \notin \pi\mathbb{Q}$ and $p, q \in \mathcal{P}_d$. For any $c \in \mathbb{C}$, if $|p(c+re^{i\alpha_1})| = |q(c+re^{i\alpha_1})|$ and $|p(c+re^{i\alpha_2})| = |q(c+re^{i\alpha_2})|$ for all $r \in \mathbb{R}$, then $[p] = [q]$.*

*Proof.* Note that

$$|R_{c,1}p(re^{i\alpha_1})| = |p(c + re^{i\alpha_1})| = |q(c + re^{i\alpha_1})| = |R_{c,1}q(re^{i\alpha_1})|$$

and

$$|R_{c,1}p(re^{i\alpha_1})| = |p(c + re^{i\alpha_1})| = |q(c + re^{i\alpha_1})| = |R_{c,1}q(re^{i\alpha_1})|$$

so by the previous theorem $[R_{c,1}p] = [R_{c,1}q]$. Recall that, because $R_{c,1}$ is a bijective linear operator, the map $[p] \mapsto [R_{c,1}p]$ is a well-defined bijection, so $[p] = [q]$. $\square$

A properly chosen Möbius transformation will allow this result concerning injectivity of the magnitude map intersecting lines to be applied to intersecting circles. Because the magnitudes on any circle can be recovered using phaseless measurements, this will give the intended result.

**Theorem 2.2.3.** *Given $\alpha \in \mathbb{R}\backslash\pi\mathbb{Q}$ and $\omega_d = e^{\frac{2i\pi}{d}}$, let $c = \frac{\omega_d+1}{2}\left(1 - \tan(\frac{\pi}{d})\cot(\alpha)\right)$. Define the set $\mathcal{S} = \{(f,g) \in \mathcal{T}_d \oplus \mathcal{T}_d : f(1) = g(-\frac{c-1}{|c-1|})$ and $f(\omega_d) = g(-\frac{c-\omega_d}{|c-\omega_d|})\}$. If the map $V_\alpha : \mathcal{P}_d/\mathbb{T} \to \mathcal{S}$ is defined by $V_\alpha([p]) = (|p|^2, |R_{c,|c-1|}p|^2)$, then $V_\alpha$ is injective.*

*Proof.* First, we show that the given map $V_\alpha$ is well defined. To do this, we must show that for any $[p] \in \mathcal{P}_d/\mathcal{T}$, $(|p|^2, |R_{c,|c-1|}p|^2) \in \mathcal{S}$. Note that the circle with center $c$ and radius $|c-1|$ passes through 1. Because the center of this circle is on the line defined by all real multiples of $\omega_d + 1$, and $\omega_d$ is the reflection of 1 across this line, the circle with center $c$ and radius $|c-1|$ passes through $\omega_d$. Thus $|c-1| = |c-\omega_d|$, and for any $p \in \mathcal{P}_d$,

$$|R_{c,|c-1|}p|^2(-\tfrac{c-1}{|c-1|}) = |R_{c,|c-1|}p(-\tfrac{c-1}{|c-1|})|^2 = |p(1)|^2 = |p|^2(1)$$

and

$$|R_{c,|c-1|}p|^2(-\tfrac{c-\omega_d}{|c-\omega_d|}) = |R_{c,|c-1|}p(-\tfrac{c-\omega_d}{|c-\omega_d|})|^2 = |p(\omega_d)|^2 = |p|^2(\omega_d)\,.$$

We define the Möbius transformation $\gamma : \mathbb{C}\backslash\{1\} \to \mathbb{C}\backslash\{-1\}$ by $\gamma(z) = \frac{1+z}{1-z}$. The image of any circle that passes through 1 under this map will be a line, so $\gamma(\mathbb{T})$ and $\gamma(c+|c-1|\mathbb{T})$ are both lines. Additionally, the point $\omega_d$ is on both circles $\mathbb{T}$ and $c+|c-1|\mathbb{T}$, so the point $\gamma(\omega_d)$ is the intersection point of the lines $\gamma(\mathbb{T})$ and $\gamma(c+|c-1|\mathbb{T})$. By conformality of Möbius transformations, the angle of intersection of the lines $\gamma(\mathbb{T})$ and $\gamma(c+|c-1|\mathbb{T})$ at

$\gamma(\omega_d)$ is equal to the angle of intersection of the circles $\mathbb{T}$ and $c+|c-1|\mathbb{T}$ at $\omega_d$.

To determine the angle of intersection of the circles $\mathbb{T}$ and $c+|c-1|\mathbb{T}$ at $\omega_d$, we will first determine the angle of the tangent lines to each circle at $\omega_d$. The tangent line to $\mathbb{T}$ at $\omega_d$ is paramentrized by $\omega_d + iu\omega_d$ for $u \in \mathbb{R}$, so the angle of this line is $\arg(i\omega_d) = \frac{\pi}{2} + \frac{2\pi}{d}$. The tangent line to $c+|c-1|\mathbb{T}$ at $\omega_d$ is parameterized by $\omega_d + iu(\omega_d - c)$ for $u \in \mathbb{R}$, so the angle of this line is

$$\arg(i(\omega_d - c)) = \frac{\pi}{2} + \arg\left(\omega_d - \frac{\omega_d+1}{2}\left(1 - \tan(\tfrac{\pi}{d})\cot(\alpha)\right)\right)$$

which may be simplified using the identity $(\omega_d+1)\tan(\frac{\pi}{d}) = -i(\omega_d - 1)$, so that

$$\begin{aligned}\arg(i(\omega_d - c)) =& \frac{\pi}{2} + \arg\left(\frac{\omega_d - 1}{2}\left(1 - i\cot(\alpha)\right)\right)\\ =& \frac{\pi}{2} + \arg\left(\frac{\omega_d - 1}{2}\right) - \alpha\\ =& \pi + \frac{\pi}{d} - \alpha\,.\end{aligned}$$

Then the angle of intersection of the circles $\mathbb{T}$ and $c+|c-1|\mathbb{T}$ at $\omega_d$ is the difference of these two angles, $\frac{\pi}{2} - \frac{\pi}{d} - \alpha$. This angle is irrational if and only if $\alpha$ is irrational, which was assumed.

We define a linear transformation $W : \mathcal{P}_d \to \mathcal{P}_d$ such that $Wp$ is the polynomial represented by the map $z \mapsto (1+z)^{d-1}p(-\frac{1-z}{1+z})$. Note that $\gamma^{-1}(z) = -\frac{1-z}{1+z}$, so $Wp(z) = 0$ only when $z = -1$ or $p(\gamma^{-1}(z)) = 0$. This only gives finitely many zeros if $p \neq 0$, so $W$ is a bijective map. Consider the nonlinear map $\tilde{V}_\alpha : \mathcal{P}_d/\mathbb{T} \to C(\gamma(\mathbb{T})) \oplus C(\gamma(c+|c-1|\mathbb{T}))$ defined such that for all $z_1 \in \gamma(\mathbb{T})$ and $z_2 \in \gamma(c+|c-1|\mathbb{T})$,

$$\left(\tilde{V}_\alpha([p])\right)(z_1, z_2) = \left(\frac{|p(z_1)|^2}{|1+z_1|^{2(d-1)}}, \frac{|p(z_2)|^2}{|1+z_2|^{2(d-1)}}\right)\,.$$

Because the lines $\gamma(\mathbb{T})$ and $\gamma(c+|c-1|\mathbb{T})$ intersect at an irrational angle, corollary 2.2.2 shows that $\tilde{V}_\alpha$ is injective. Thus, the map $[p] \mapsto \tilde{V}_\alpha[Wp]$ is injective.

Note that by the definition of $W$, if $[p] \in \mathcal{P}_d/\mathbb{T}$ then for any $z_1 \in \gamma(\mathbb{T})$ and any $z_2 \in \gamma(c+|c-1|\mathbb{T})$,

$$\begin{aligned}\left(\tilde{V}_\alpha([Wp])\right)(z_1,z_2) &= \left(\frac{|Wp(z_1)|^2}{|1+z_1|^{2(d-1)}}, \frac{|Wp(z_2)|^2}{|1+z_2|^{2(d-1)}}\right) \\ &= \left(\left|p\left(\gamma^{-1}(z_1)\right)\right|^2, \left|p\left(\gamma^{-1}(z_2)\right)\right|^2\right) \\ &= \left(|p|^2\left(\gamma^{-1}(z_1)\right), |R_{c,|c-1|}p|^2\left(\frac{\gamma^{-1}(z_2)-c}{|c-1|}\right)\right).\end{aligned}$$

Consider the nonlinear map $\tilde{W}_c : \mathcal{S} \to C(\gamma(\mathbb{T})) \oplus C(\gamma(c+|c-1|\mathbb{T}))$ defined such that if $(f,g) \in \mathcal{S}$, then for all $z_1 \in \gamma(\mathbb{T})$ and $z_2 \in \gamma(c+|c-1|\mathbb{T})$,

$$\left(\tilde{W}_c(f,g)\right)(z_1,z_2) = \left(f\left(\gamma^{-1}(z_1)\right), g\left(\frac{\gamma^{-1}(z_2)-c}{|c-1|}\right)\right).$$

With this definition, we see that for any $[p] \in \mathcal{P}_d/\mathbb{T}$, $\tilde{V}_\alpha[Wp] = \tilde{W}_c V_\alpha[p]$. Thus the composition $\tilde{W}_c V_\alpha$ is injective, and so $V_\alpha$ is injective. $\square$

Given the injective map in the above theorem, phaseless measurements in the form of a basis for the space $\mathcal{S}$ are sufficient to determine $[p]$ uniquely.

**Corollary 2.2.4.** *Given $\alpha \in \mathbb{R}\backslash\pi\mathbb{Q}$ and $\omega_d = e^{\frac{2i\pi}{d}}$, let $c = \frac{\omega_d+1}{2}\left(1-\tan(\frac{\pi}{d})\cot(\alpha)\right)$. Let $\omega_{2d-1} = e^{\frac{2i\pi}{2d-1}}$ be the $2d-1$st root of unity. For any $p \in \mathcal{P}_d$, the values $\left\{|p(\omega_{2d-1}^k)|^2\right\}_{k=2}^{2d-1}$ and $\left\{\left|p\left(c+|c-1|\omega_{2d-1}^j\right)\right|^2\right\}_{j=0}^{2d-1}$ are sufficient to determine $[p]$ uniquely.*

## 2.3 An algorithm for phase retrieval in $6d-3$ measurements

Although there exist injective sets of measurements using maps $[p] \mapsto (|p|^2, |Tp|^2)$ for some choice of linear operator $T$, no corresponding stable algorithm with which to reconstruct the signal $[p]$ has been discovered. To provide a solution for the phase retrieval problem using a small number of measurements, we shall provide a stable algorithm with which to recover $[p]$ from $(|p|^2, |(I - R_{0,\omega_d})p|^2, |(I - iR_{0,\omega_d})p|^2)$.

Given $p \in \mathcal{P}_d$, the phase retrieval algorithm that we shall use proceeds in three steps:

**Step 1.** First, we augment a finite number of magnitude measurements to an infinite family of such measurements. For each of $2d-1$ equally spaced points $\{\omega_{2d-1}^j\}_{j=0}^{2d-1}$ on $\mathbb{T}$, we sample $|p(\omega_{2d-1}^j)|^2$, $|p(\omega_{2d-1}^j) - p(\omega_{2d-1}^j\omega_d)|^2$, and $|p(\omega_{2d-1}^j) - ip(\omega_{2d-1}^j\omega_d)|^2$. Then we use the Dirichlet kernel to interpolate these measurements to obtain three trigonometric polynomials: $|p|^2$, $|(I-R_{0,\omega_d})p|^2$, and $|(I-iR_{0,\omega_d})p|^2$. In the presence of noise, we get approximating trigonometric polynomials $f_0 \approx |p|^2$, $f_1 \approx |(I-R_{0,\omega_d})p|^2$, and $f_2 \approx |(I - iR_{0,\omega_d})p|^2$.

**Step 2.** We select suitable non-zero magnitude measurements. To do this we choose a $z_0 \in \mathbb{T}$, such that $\min_j |p(\omega_d^j z_0)|^2$ is maximized. A lemma in appendix A.1 will be used to obtain a lower bound for this maximum that only depends on the dimension $d$ and the norm of $p$. Then we sample values of each of the three polynomials from step 1 at the points $\{z_0\omega_d^j\}_{j=0}^{d-1}$. These values are approximations for the magnitudes of the components of $p$, $(I - R_{0,\omega_d})p$, and $(I - iR_{0,\omega_d})p$ with respect to the basis $\{\frac{1}{\sqrt{d}}K_{z_0\omega_d^j}\}_{j=0}^{d-1}$. We know that the components of $f_0$ are not zero, by our choice of $z_0$.

**Step 3.** For any vector $x \in \mathbb{C}^d$ and any basis $\{e_j\}_{j=0}^{d-1}$, we have the relation

$$\begin{aligned}\overline{\langle x, e_j\rangle}\langle x, e_{j+1}\rangle =& \tfrac{1}{2}\left((1-i)\left(|\langle x, e_j\rangle|^2 + |\langle x, e_{j+1}\rangle|^2\right)\right. \\ &\left. -|\langle x, e_j - e_{j+1}\rangle|^2 + i|\langle x, e_j - ie_{j+1}\rangle|^2\right)\end{aligned}$$

If we consider the basis set $\{\frac{1}{\sqrt{d}}K_{z_0\omega_d^j}\}_{j=0}^{d-1}$, then we get that

$$\begin{aligned}&\overline{\langle p, K_{z_0\omega_d^j}\rangle}\langle p, K_{z_0\omega_d^{j+1}}\rangle \\ &\qquad = \tfrac{1}{2}\left((1-i)\left(|p|^2(z_0\omega_d^j) + |p|^2(z_0\omega_d^j)\right)\right. \\ &\qquad\qquad \left. -|(I - R_{0,\omega_d})p|^2(z_0\omega_d^j) + i|(I - iR_{0,\omega_d})p|^2(z_0\omega_d^j)\right) .\end{aligned}$$

This phase relationship is a linear combination of sample values obtained in step 2. Because all of the values $\langle p, K_{z_0\omega_d^j}\rangle$ are bounded away from 0, we know that this phase relation will always be nonzero. Thus, with the base case of assuming that $\langle p, K_{z_0}\rangle$ is real, we may inductively obtain the phase of consecutive coefficients with respect to this basis using the relation

$$\langle p, K_{z_0\omega_d^{j+1}}\rangle = \frac{\langle p, K_{z_0\omega_d^j}\rangle\left(\overline{\langle p, K_{z_0\omega_d^j}\rangle}\langle p, K_{z_0\omega_d^{j+1}}\rangle\right)}{|\langle p, K_{z_0\omega_d^j}\rangle|^2} .$$

These coefficients with respect to this basis are sufficient to recover the signal.

The time complexity of this algorithm is polynomial. Step 1 requires the addition of $d$ different $d$ dimensional terms, so step 1 has time complexity $d^2$. Step 2 requires you to find a point such that a set of $d$ points a have values that are bounded away from zero. Because a polynomial in $\mathcal{P}_d$ has at most $d-1$ roots, $d$ different sets of $d$ points will guarantee that at least one set is bounded away from zero. Because polynomial evaluation has time complexity $d$, and we are evaluating at $d^2$ points, step 2 has time complexity $d^3$. Step 3 consists of inductively multiplying a starting value by a stored value $d$ times, so step 3 has time complexity $d$. Thus, this algorithm can run in polynomial time.

In the presence of error, the trigonometric polynomials obtained in step 1 are perturbed. This error carries through to step 3. Section 2.5 will provide an upper bound for the size of the perturbation of the trigonometric polynomials. We shall explore step 3 in more detail in the next section, including an examination of propagation of the error in step 3. The algorithm can be shown to recover the signal perfectly in the case without noise by setting the noise to 0 and observing that the error bound then also becomes 0.

## 2.4 Recovery of full vectors in the presence of noise

In this section we explore a generalized version of the third step of the algorithm presented in section 2.3. Instead of using the particular basis given in step 3 of the algorithm, we show that the third step of the algorithm will give stability results for phase retrieval for any basis in which we wish to recover vectors that have no zero coefficients with respect to the chosen basis. The measurements and algorithm used here is similar to that used by Flammia, Silverfarb, and Caves [38], in that it uses a polarization like identity in order to obtain a phase relation. However, our choice of measurements is such that these stability results apply to the phase propagation portion of the algorithm presented in section 2.3.

**Definition 2.4.1.** If $\{e_j\}_{j=0}^{d-1}$ is a basis for $\mathbb{C}^d$, and $x \in \mathbb{C}^d$, then we say that $x$ is full with respect to $\{e_j\}_{j=0}^{d-1}$ if for all $j$ from 0 to $d-1$, $\langle x, e_j \rangle \neq 0$.

Recall that phase recovery of full vectors requires $3d-2$ phaseless measurements. Given a vector $x \in \mathbb{C}^d$ and any basis $\{e_j\}_{j=0}^{d-1}$ for $\mathbb{C}^d$ in which the vector $x$ is full, the phase retrieval algorithm proceeds as follows

**Generalized Step 3.** We are given the phaseless measurements

$$\begin{aligned}\{|\langle x, e_k\rangle|^2\}_{k=0}^{d-1}\\ \{|\langle x, e_k - e_{k+1}\rangle|^2\}_{k=0}^{d-2}\\ \{|\langle x, e_k - ie_{k+1}\rangle|^2\}_{k=0}^{d-2}.\end{aligned}$$

Using these values, we have the relation

$$\begin{aligned}\overline{\langle x, e_j\rangle}\langle x, e_{j+1}\rangle =& \tfrac{1}{2}\left((1-i)\left(|\langle x, e_j\rangle|^2 + |\langle x, e_{j+1}\rangle|^2\right)\right.\\ &\left. -|\langle x, e_j - e_{j+1}\rangle|^2 + i|\langle x, e_j - ie_{j+1}\rangle|^2\right)\end{aligned}$$

Because all of the values $\langle x, e_j\rangle$ are bounded away from 0, we know that this phase relation will always be nonzero. Thus, with the base case of assuming that $\langle x, e_0\rangle$ is real and positive, we may inductively obtain the phase of consecutive coefficients with respect to this basis using the relation

$$\langle x, e_{j+1}\rangle = \frac{\langle x, e_j\rangle\left(\overline{\langle x, e_j\rangle}\langle x, e_{j+1}\rangle\right)}{|\langle x, e_j\rangle|^2}.$$

These coefficients with respect to this basis are sufficient to recover the signal.

We want to explore the impact of noise on the above algorithm. Thus, if we have measurements

$$\begin{aligned}\{|\langle x, e_k\rangle|^2 + \epsilon_k\}_{k=0}^{d-1}\\ \{|\langle x, e_k - e_{k+1}\rangle|^2 + \epsilon_{k+d}\}_{k=0}^{d-2}\\ \{|\langle x, e_k - ie_{k+1}\rangle|^2 + \epsilon_{k+2d-1}\}_{k=0}^{d-2},\end{aligned}$$

for some $\epsilon \in \mathbb{R}^{3d-2}$, we want to show that the recovered vector is close to the signal. First, we shall show that the error in each component depends linearly on $\epsilon$.

**Lemma 2.4.2.** *Let $(m_j)_{j=0}^{d-1} \in (0,1]^d$. For any vectors $x \in \mathbb{C}^d$ and $\epsilon \in \mathbb{R}^{3d-2}$, define constants $C \in \mathbb{R}$ and $\tilde{\epsilon} \in \mathbb{R}^d$ such that $C = \frac{(1+\sqrt{2})\|\epsilon\|_\infty + \|x\|_\infty^2}{\min(m)\|x\|_\infty^2}$ and for all $j$ from $0$ to $d-1$, $\tilde{\epsilon}_j = \frac{1}{2}\left(\sqrt{2}|\epsilon_j| + \sqrt{2}|\epsilon_{j+1}| + |\epsilon_{d+j}| + |\epsilon_{2d+j-1}|\right)$. If $\{e_j\}_{j=0}^{d-1}$ is an orthonormal basis such that $|\langle x, e_0\rangle|^2 \geq m_0\|x\|_\infty^2$, and for all $j$ from $0$ to $d-1$, $|\langle x, e_j\rangle|^2 + \epsilon_j \geq m_j\|x\|_\infty^2$, then a vector $y$ may be obtained such that for all $k$ from $0$ to $d-1$,*

$$\left|y_k - \frac{\overline{x_0}}{|x_0|}x_k\right| \leq \left(\sum_{j=0}^{k-1}\left(C^{k-1-j}\frac{\tilde{\epsilon_j} + |\epsilon_j|}{m_j}\right) + \frac{C^k|\epsilon_0|}{2\sqrt{m_0}}\right)\frac{1}{\|x\|_\infty}$$

*by using the algorithm presented above with the phaseless measurements*

$$\begin{aligned}\{|\langle x, e_k\rangle|^2 + \epsilon_k\}_{k=0}^{d-1} \\ \{|\langle x, e_k - e_{k+1}\rangle|^2 + \epsilon_{k+d}\}_{k=0}^{d-2} \\ \{|\langle x, e_k - ie_{k+1}\rangle|^2 + \epsilon_{k+2d-1}\}_{k=0}^{d-2}.\end{aligned}$$

*Proof.* The proof proceeds by induction on $k$. For the base case, let $y_0 = \sqrt{|\langle x, e_0\rangle|^2 + \epsilon_0}$. Then by the mean value theorem and concavity of the square root, there exists a $\xi$ between $|\langle x, e_0\rangle|^2 + \epsilon_0$ and $|\langle x, e_0\rangle|^2$ (so that $\xi \geq m_0\|x\|_\infty^2 > 0$) such that

$$\begin{aligned}\left|y_0 - \frac{\overline{x_0}}{|x_0|}x_0\right| &= \left|\sqrt{|\langle x, e_0\rangle|^2 + \epsilon_0} - \sqrt{|\langle x, e_0\rangle|^2}\right| \\ &= \frac{|\epsilon_0|}{2\sqrt{\xi}} \\ &\leq \frac{|\epsilon_0|}{2\sqrt{m_0}\|x\|_\infty}\end{aligned}$$

For the $k$th (with $k < d-1$) inductive step, we assume that we have obtained $y_k$ with the given information such that $E_k = \left| y_k - \frac{\overline{x_1}}{|x_1|} x_k \right| \leq \left( \sum_{j=0}^{k-1} \left( C^{k-1-j} \frac{|\tilde{\epsilon_j}| + |\epsilon_j|}{m_j} \right) + \frac{C^k |\epsilon_0|}{2\sqrt{m_0}} \right) \frac{1}{\|x\|_\infty}$. Note that

$$\overline{x_k} x_{k+1} = \frac{1}{2} \left( (1-i)|x_k|^2 + (1-i)|x_{k+1}|^2 - |x_k - x_{k+1}|^2 + i|x_k - ix_{k+1}|^2 \right)$$

so an approximation for $\overline{x_k} x_{k+1}$ may be obtained as a linear combination of the perturbed measurements. If we let

$$\begin{aligned} t_k =& \frac{1}{2} \left( (1-i) \left( |\langle x, e_k \rangle|^2 + \epsilon_k + |\langle x, e_{k+1} \rangle|^2 + \epsilon_{k+1} \right) \right. \\ & \left. - \left( |\langle x, e_k - e_{k+1} \rangle|^2 + \epsilon_{k+d} \right) + i \left( |\langle x, e_k - ie_{k+1} \rangle|^2 + \epsilon_{k+2d-1} \right) \right) \end{aligned}$$

and

$$y_{k+1} = \frac{t_k}{|\langle x, e_k \rangle|^2 + \epsilon_k} y_k \,,$$

then $t_k \approx \overline{x_k} x_{k+1}$ and $y_{k+1} \approx \frac{\overline{x_k} x_{k+1}}{|\langle x, e_k \rangle|^2 + \epsilon_k} y_k$. A direct computation shows the error for the approximation of the term used in phase propagation,

$$\begin{aligned} |t_k - \overline{x_k} x_{k+1}| =& \left| \frac{(1-i)(|x_k|^2 + \epsilon_k) + (1-i)(|x_{k+1}|^2 + \epsilon_{k+1})}{2} \right. \\ & - \frac{(|x_k - x_{k+1}|^2 + \epsilon_{d+k}) - i(|x_k - ix_{k+1}|^2 + \epsilon_{2d+k-1})}{2} \\ & - \frac{(1-i)|x_k|^2 + (1-i)|x_{k+1}|^2}{2} \\ & \left. + \frac{|x_k - x_{k+1}|^2 - i|x_k - ix_{k+1}|^2}{2} \right| \\ =& \left| \frac{(1-i)\epsilon_k + (1-i)\epsilon_{k+1} - \epsilon_{d+k} + i\epsilon_{2d+k-1}}{2} \right| \\ \leq& \frac{\sqrt{2}|\epsilon_k| + \sqrt{2}|\epsilon_{k+1}| + |\epsilon_{d+k}| + |\epsilon_{2d+k-1}|}{2} \\ =& \tilde{\epsilon}_k \end{aligned}$$

which satisfies $\tilde{\epsilon_k} \leq (1+\sqrt{2})\|\epsilon\|_\infty$. We use similar calculations to simplify the relationship between the vector and approximate recovery,

$$\begin{aligned}
\left|y_{k+1} - \frac{\overline{x_0}}{|x_0|}x_{k+1}\right| &= \left|\frac{t_k}{|\langle x, e_k\rangle|^2 + \epsilon_k}y_k - \frac{\overline{x_k}x_{k+1}}{|x_k|^2}\frac{\overline{x_1}}{|x_1|}x_k\right| \\
&= \left|\frac{|x_k|^2 t_k y_k - \overline{x_k}x_{k+1}\frac{\overline{x_1}}{|x_1|}x_k(|x_k|^2+\epsilon_k)}{(|x_k|^2+\epsilon_k)|x_k|^2}\right| \\
&= \left|\frac{|x_k|^2(t_k - \overline{x_k}x_{k+1})y_k + |x_k|^2\overline{x_k}x_{k+1}(y_k - \frac{\overline{x_1}}{|x_1|}x_k) - \overline{x_k}x_{k+1}\frac{\overline{x_1}}{|x_1|}x_k\epsilon_k}{(|x_k|^2+\epsilon_k)|x_k|^2}\right| \\
&= \left|\frac{(t_k - \overline{x_k}x_{k+1})y_k + \overline{x_k}x_{k+1}(y_k - \frac{\overline{x_1}}{|x_1|}x_k) - x_{k+1}\frac{\overline{x_1}}{|x_1|}\epsilon_k}{|x_k|^2+\epsilon_k}\right|.
\end{aligned}$$

We use the assumption that $|\langle x, e_k\rangle|^2 + \epsilon_k \geq m_k\|x\|_\infty^2$ to get that

$$\left|y_{k+1} - \frac{\overline{x_0}}{|x_0|}x_{k+1}\right| \leq \frac{\left|(t_k - \overline{x_k}x_{k+1})y_k + \overline{x_k}x_{k+1}(y_k - \frac{\overline{x_1}}{|x_1|}x_k) - x_{k+1}\frac{\overline{x_1}}{|x_1|}\epsilon_k\right|}{m_k\|x\|_\infty^2}.$$

Next, we estimate using the triangle inequality

$$\begin{aligned}
\left|y_{k+1} - \frac{\overline{x_0}}{|x_0|}x_{k+1}\right| &\leq \frac{|t_k - \overline{x_k}x_{k+1}|\,|y_k| + |\overline{x_k}x_{k+1}||y_k - \frac{\overline{x_1}}{|x_1|}x_k| + |x_{k+1}\frac{\overline{x_1}}{|x_1|}\epsilon_k|}{m_k\|x\|_\infty^2} \\
&\leq \frac{|t_k - \overline{x_k}x_{k+1}|\,(|y_k - \frac{\overline{x_1}}{|x_1|}x_k| + |x_k|) + |\overline{x_k}x_{k+1}||y_k - \frac{\overline{x_1}}{|x_1|}x_k| + |x_{k+1}\epsilon_k|}{m_k\|x\|_\infty^2}.
\end{aligned}$$

Recalling that $E_k = |y_k - \frac{\overline{x_1}}{|x_1|}x_k|$ was bounded by the induction assumption, we get

$$\begin{aligned}
\left|y_{k+1} - \frac{\overline{x_0}}{|x_0|}x_{k+1}\right| &\leq \frac{\tilde{\epsilon_k}(E_k + \|x\|_\infty) + \|x\|_\infty^2 E_k + \|x\|_\infty|\epsilon_k|}{m_k\|x\|_\infty^2} \\
&= \frac{\tilde{\epsilon_k} + |\epsilon_k|}{m_k\|x\|_\infty^2}\|x\|_\infty + \frac{(1+\sqrt{2})\|\epsilon\|_\infty + \|x\|_\infty^2}{m_k\|x\|_\infty^2}E_k
\end{aligned}$$

and using the definition of $C$ to simplify gives

$$\left| y_{k+1} - \frac{\overline{x_0}}{|x_0|} x_{k+1} \right| \leq \frac{\tilde{\epsilon_k} + |\epsilon_k|}{m_k \|x\|_\infty} + C E_k \,.$$

Finally, we replace all occurrences of $E_k$ with the value given by the inductive assumption, and simplify

$$\begin{aligned}
\left| y_{k+1} - \frac{\overline{x_0}}{|x_0|} x_{k+1} \right| &\leq \frac{\tilde{\epsilon_k} + |\epsilon_k|}{m_k \|x\|_\infty} + C \left( \sum_{j=0}^{k-1} \left( C^{k-1-j} \frac{\tilde{\epsilon_j} + |\epsilon_j|}{m_j} \right) + \frac{C^k |\epsilon_0|}{2\sqrt{m_0}} \right) \frac{1}{\|x\|_\infty} \\
&\leq \frac{\tilde{\epsilon_k} + |\epsilon_k|}{m_k \|x\|_\infty} + \left( \sum_{j=0}^{k-1} \left( C^{k-j} \frac{\tilde{\epsilon_j} + |\epsilon_j|}{m_j} \right) + \frac{C^{k+1} |\epsilon_0|}{2\sqrt{m_0}} \right) \frac{1}{\|x\|_\infty} \\
&= \left( \sum_{j=0}^{k} \left( C^{k-j} \frac{\tilde{\epsilon_j} + |\epsilon_j|}{m_j} \right) + \frac{C^{k+1} |\epsilon_0|}{2\sqrt{m_0}} \right) \frac{1}{\|x\|_\infty} \,.
\end{aligned}$$

□

Note that if each component of $\epsilon$ is 0, then the error is 0 in each component, so this algorithm reconstructs $x$ perfectly. In the next lemma, we show a bound on the $\ell_1$ norm of the difference between the recovered vector and the input signal.

**Lemma 2.4.3.** *Let $(m_j)_{j=0}^{d-1} \in (0,1]^d$. For any vectors $x \in \mathbb{C}^d$ and $\epsilon \in \mathbb{R}^{3d-2}$, define $C \in \mathbb{R}$ such that $C = \frac{(1+\sqrt{2})\|\epsilon\|_\infty + \|x\|_\infty^2}{\min(m)\|x\|_\infty^2}$. If $\{e_j\}_{j=0}^{d-1}$ is an orthonormal basis such that $|\langle x, e_0 \rangle|^2 \geq m_0 \|x\|_\infty^2$, and for all $j$ from 0 to $d-1$, $|\langle x, e_j \rangle|^2 + \epsilon_j \geq m_j \|x\|_\infty^2$, then a vector $y$ may be obtained such that*

$$\sum_{k=1}^{d} \left| y_k - \frac{\overline{x_1}}{|x_1|} x_k \right| \leq \|w(C, \min(m))\|_2 \frac{\|\epsilon\|_2}{\|x\|_\infty}$$

*for a weight vector* $w(C,u) \in \mathbb{R}^{3d-2}$ *satisfying*

$$w(C,u)_j = \begin{cases} \sum_{k=1}^{d-1} \frac{C^{k-1}(1+\frac{\sqrt{2}}{2})}{u} + \sum_{k=0}^{d-1} \frac{C^k}{2\sqrt{u}} & \text{if } j=0 \\ \sum_{k=1}^{d-j-1} \frac{C^{k-1}(1+\frac{\sqrt{2}}{2})}{u} + \sum_{k=0}^{d-j-1} \frac{C^k \frac{\sqrt{2}}{2}}{u} & \text{if } 1 \le j \le d-1 \\ \sum_{k=1}^{2d-1-j} \frac{C^{k-1}}{2u} & \text{if } d \le j \le 2d-2 \\ \sum_{k=1}^{3d-2-j} \frac{C^{k-1}}{2u} & \text{if } 2d-1 \le j \le 3d-3 \end{cases}$$

*by using the algorithm presented above with the phaseless measurements*

$$\{|\langle x, e_k\rangle|^2 + \epsilon_k\}_{k=0}^{d-1}$$

$$\{|\langle x, e_k - e_{k+1}\rangle|^2 + \epsilon_{k+d}\}_{k=0}^{d-2}$$

$$\{|\langle x, e_k - ie_{k+1}\rangle|^2 + \epsilon_{k+2d-1}\}_{k=0}^{d-2}.$$

*Proof.* By Lemma 2.4.2, we know that for $k=0$,

$$\left| y_k - \frac{\overline{x_0}}{|x_0|} x_k \right| \le \frac{|\epsilon_0|}{2\sqrt{m_0}} \frac{1}{\|x\|_\infty}.$$

Also, by Lemma 2.4.2, we know that if $\tilde{\epsilon_j} = \frac{1}{2}\left(\sqrt{2}|\epsilon_j| + \sqrt{2}|\epsilon_{j+1}| + |\epsilon_{d+j}| + |\epsilon_{2d+j-1}|\right)$ for all $j$ from 0 to $d-1$, then for all $k$ from 1 to $d-1$,

$$\begin{aligned} \left| y_k - \frac{\overline{x_0}}{|x_0|} x_k \right| &\le \left( \sum_{j=0}^{k-1} \left( C^{k-1-j} \frac{\tilde{\epsilon_j} + |\epsilon_j|}{m_j} \right) + \frac{C^k |\epsilon_0|}{2\sqrt{m_0}} \right) \frac{1}{\|x\|_\infty} \\ &= \left( \frac{C^{k-1}(1+\frac{\sqrt{2}}{2})}{m_0} + \frac{C^k}{2\sqrt{m_0}} \right) \frac{|\epsilon_0|}{\|x\|_\infty} \\ &\quad + \sum_{j=1}^{k-1} \left( \frac{C^{k-1-j}(1+\frac{\sqrt{2}}{2})}{m_j} + \frac{C^{k-j}\frac{\sqrt{2}}{2}}{m_{j-1}} \right) \frac{|\epsilon_j|}{\|x\|_\infty} \\ &\quad + \frac{\sqrt{2}}{2m_{k-1}} \frac{|\epsilon_k|}{\|x\|_\infty} + \sum_{j=0}^{k-1} \frac{C^{k-1-j}}{2m_j} \frac{|\epsilon_{d+j}|}{\|x\|_\infty} + \sum_{j=0}^{k-1} \frac{C^{k-1-j}}{2m_j} \frac{|\epsilon_{2d+j-1}|}{\|x\|_\infty}. \end{aligned}$$

When we sum these terms over all $k$, we get

$$\sum_{k=0}^{d-1}\left|y_k-\frac{\overline{x_0}}{|x_0|}x_k\right| \leq \left(\frac{1}{2\sqrt{m_0}}+\sum_{k=1}^{d-1}\left(\frac{C^{k-1}(1+\frac{\sqrt{2}}{2})}{m_0}+\frac{C^k}{2\sqrt{m_0}}\right)\right)\frac{|\epsilon_0|}{\|x\|_\infty}$$
$$+\sum_{k=1}^{d-1}\sum_{j=1}^{k-1}\left(\frac{C^{k-1-j}(1+\frac{\sqrt{2}}{2})}{m_j}+\frac{C^{k-j}\frac{\sqrt{2}}{2}}{m_{j-1}}\right)\frac{|\epsilon_j|}{\|x\|_\infty}$$
$$+\sum_{k=1}^{d-1}\frac{\sqrt{2}}{2m_{k-1}}\frac{|\epsilon_k|}{\|x\|_\infty}+\sum_{k=1}^{d-1}\sum_{j=0}^{k-1}\frac{C^{k-1-j}}{2m_j}\frac{|\epsilon_{d+j}|}{\|x\|_\infty}+\sum_{k=1}^{d-1}\sum_{j=0}^{k-1}\frac{C^{k-1-j}}{2m_j}\frac{|\epsilon_{2d+j-1}|}{\|x\|_\infty}$$

and by rearranging

$$\sum_{k=0}^{d-1}\left|y_k-\frac{\overline{x_0}}{|x_0|}x_k\right| = \left(\sum_{k=1}^{d-1}\frac{C^{k-1}(1+\frac{\sqrt{2}}{2})}{m_0}+\sum_{k=0}^{d-1}\frac{C^k}{2\sqrt{m_0}}\right)\frac{|\epsilon_0|}{\|x\|_\infty}$$
$$+\sum_{j=1}^{d-2}\sum_{k=j+1}^{d-1}\left(\frac{C^{k-1-j}(1+\frac{\sqrt{2}}{2})}{m_j}+\frac{C^{k-j}\frac{\sqrt{2}}{2}}{m_{j-1}}\right)\frac{|\epsilon_j|}{\|x\|_\infty}$$
$$+\sum_{j=1}^{d-1}\frac{\sqrt{2}}{2m_{j-1}}\frac{|\epsilon_j|}{\|x\|_\infty}$$
$$+\sum_{j=0}^{d-2}\sum_{k=j+1}^{d-1}\frac{C^{k-1-j}}{2m_j}\frac{|\epsilon_{d+j}|}{\|x\|_\infty}+\sum_{j=0}^{d-2}\sum_{k=j+1}^{d-1}\frac{C^{k-1-j}}{2m_j}\frac{|\epsilon_{2d+j-1}|}{\|x\|_\infty}$$

and because every term in this sequence is non-increasing in each $m_j$

$$\leq \sum_{j=0}^{3d-3} w(C,\min(m))_j\frac{|\epsilon_j|}{\|x\|_\infty}$$
$$=\frac{1}{\|x\|_\infty}\sum_{j=1}^{3d-2} w(C,\min(m))_j\frac{\bar{\epsilon}_j}{|\epsilon_j|}\epsilon_j\,.$$

Then by the Cauchy-Schwarz inequality, we get that

$$\sum_{k=1}^{d}\left|y_k-\frac{\overline{x_1}}{|x_1|}x_k\right| \leq \|\tilde{w}\|_2\frac{\|\epsilon\|_2}{\|x\|_\infty}$$

where $\tilde{w}_j = w(C, \min(m))_j \frac{\bar{\epsilon}_j}{|\epsilon_j|}$. Note that $\|\tilde{w}\|_2 = \|w(C, \min(m))\|_2$, which proves the claim. $\square$

## 2.5 Interpolation in the presence of noise

In the presence of noise, a few lemmas are needed to show how noise propagates through Dirichlet kernel interpolation. First, we will note that the basis of point evaluations of the analytic polynomials are point evaluations at $d$ equally spaced points around $\mathbb{T}$, while the basis of point evaluations of the trigonometric polynomials are point evaluations at $2d-1$ equally spaced points around $\mathbb{T}$. The following lemma observes the effect of evaluating a trigonometric polynomial at $d$ equally spaced points around $\mathbb{T}$.

**Lemma 2.5.1.** *For any trigonometric polynomial $f : z \mapsto \sum_{k=-(d-1)}^{d-1} c_k z^k$ of degree at most $d-1$, and any $z_0 \in \mathbb{C}$ such that $|z_0| = 1$,*

$$\sum_{j=0}^{d-1} \left| \left\langle f, \frac{1}{\sqrt{d}} D_{z_0 \omega_d^j, d-1} \right\rangle \right|^2 \leq 2\|f\|_2^2$$

*Proof.* Note that the set $\{\frac{1}{\sqrt{d}} D_{z_0 \omega_d^j, d-1}\}_{j=0}^{d-1}$ is a Bessel sequence, because it is finite. Thus, the Bessel bound is equal to the operator norm of the Gram matrix. Note that for all $j$ from 0 to $d-1$,

$$\left\langle \frac{1}{\sqrt{d}} D_{z_0 \omega_d^j, d-1}, \frac{1}{\sqrt{d}} D_{z_0 \omega_d^j, d-1} \right\rangle = \frac{2d-1}{d} = 2 - \frac{1}{d}$$

and for all $l$ from 0 to $d-1$ with $j \neq l$,

$$\begin{aligned}\left\langle \frac{1}{\sqrt{d}} D_{z_0\omega_d^j, d-1}, \frac{1}{\sqrt{d}} D_{z_0\omega_d^l, d-1} \right\rangle + \frac{1}{d} =& \frac{1}{d}\left(1 + \sum_{k=-(d-1)}^{d-1} \omega_d^{(l-j)k}\right) \\ =& \frac{1}{d}\left(\sum_{k=0}^{d-1} \omega_d^{(j-l)k} + \sum_{k=0}^{d-1} \omega_d^{(l-j)k}\right) \\ =& 0\end{aligned}$$

Thus, if we let $1_d$ be the $d \times d$ matrix of all 1s, then the Gram matrix $G$ satisfies $G = 2I_d - \frac{1}{d}1_d$, and has operator norm $\|G\| = 2$. Thus the Bessel bound for this sequence is 2, and

$$\sum_{j=0}^{d-1} \left| \left\langle f, \frac{1}{\sqrt{d}} D_{z_0\omega_d^j, d-1} \right\rangle \right|^2 \leq 2\|f\|_2^2$$

□

Now we will observe the effect of noise and the effect of oversampling on Dirichlet kernel interpolation.

**Lemma 2.5.2.** *If a trigonometric polynomial $f \in \mathcal{T}_d$ is measured at $2N-1$ equally spaced points on $\mathbb{T}$, where $N \geq d$, with a noise vector $\eta \in \mathbb{R}^{2N-1}$, then the recovered $2d-1$ degree trigonometric polynomial $\tilde{f} \in \mathcal{T}_d$ will have error that satisfies*

$$\|\tilde{f} - f\|_2 \leq \frac{1}{\sqrt{2N-1}} \|\eta\|_2$$

*Proof.* Note that any trigonometric polynomial $f \in \mathcal{T}_d$ may be expressed as an element of $\mathcal{T}_N$ for any $N \geq d$. Then Dirichlet kernel interpolation may be used in this space, but should still recover $f$. Thus $f = \sum_{j=0}^{2N-2} f(\omega_{2N-1}^j) \frac{1}{2N-1} D_{\omega_{2N-1}^j, N-1}$, where $\omega_{2N-1}$ is the

$2N-1$st root of unity. Thus,

$$f = \sum_{j=0}^{2N-2} f(\omega_{2N-1}^j) \frac{1}{2N-1} D_{\omega_{2N-1}^j, N-1}$$

and if we let

$$\tilde{f}_N = \sum_{j=0}^{2N-2} (f(\omega_{2N-1}^j) + \eta_j) \frac{1}{2N-1} D_{\omega_{2N-1}^j, N-1}$$

then

$$\begin{aligned} \|\tilde{f}_N - f\|_2^2 &= \left\| \sum_{j=0}^{2N-2} \eta_j \frac{1}{2N-1} D_{\omega_{2N-1}^j, N-1} \right\|_2^2 \\ &= \sum_{j=0}^{2N-2} \left| \eta_j \frac{1}{\sqrt{2N-1}} \right|^2 \\ &= \frac{1}{2N-1} \|\eta\|_2^2 . \end{aligned}$$

Note that $f$ has degree at most $2d-1$, even when interpolated as a $2N-1$ degree trigonometric polynomial, so if the extra dimensions of $\tilde{f}_N$ are truncated, and we let

$$\tilde{f} = \sum_{j=0}^{2N-2} (f(\omega_{2N-1}^j) + \eta_j) \frac{1}{2N-1} D_{\omega_{2N-1}^j, d-1}$$

then we get that

$$\|\tilde{f} - f\|_2 \leq \|\tilde{f}_N - f\|_2 = \frac{1}{\sqrt{2N-1}} \|\eta\|_2$$

$\square$

We also observe the mean squared error of the effect of noise and the effect of oversampling on Dirichlet kernel interpolation.

**Lemma 2.5.3.** *If a trigonometric polynomial $f \in \mathcal{T}_d$ is measured at $2N-1$ equally spaced* *points on $\mathbb{T}$, where $N \geq d$, with a noise vector $\eta \in \mathbb{R}^{2N-1}$ composed of i.i.d. Gaussian entries, then the recovered $2d-1$ degree trigonometric polynomial $\tilde{f} \in \mathcal{T}_d$ will have error that satisfies*

$$\mathbb{E}[\|\tilde{f} - f\|_2^2] = \frac{2d-1}{(2N-1)^2}\mathbb{E}[\|\eta\|_2^2]$$

*Proof.* Note that any trigonometric polynomial $f \in \mathcal{T}_d$ may be expressed as an element of $\mathcal{T}_N$ for any $N \geq d$. Then Dirichlet kernel interpolation may be used in this space, but should still recover $f$. Thus $f = \sum_{j=0}^{2N-2} f(\omega_{2N-1}^j)\frac{1}{2N-1}D_{\omega_{2N-1}^j, N-1}$, where $\omega_{2N-1}$ is the $2N-1$st root of unity. Thus,

$$f = \sum_{j=0}^{2N-2} f(\omega_{2N-1}^j)\frac{1}{2N-1}D_{\omega_{2N-1}^j, N-1}$$

and if we let

$$\tilde{f}_N = \sum_{j=0}^{2N-2} (f(\omega_{2N-1}^j) + \eta_j)\frac{1}{2N-1}D_{\omega_{2N-1}^j, N-1}$$

then

$$\begin{aligned}
\mathbb{E}[\|\tilde{f}_N - f\|_2^2] =& \mathbb{E}\left[\left\|\sum_{j=0}^{2N-2} \eta_j \frac{1}{2N-1}D_{\omega_{2N-1}^j, N-1}\right\|_2^2\right] \\
=& \mathbb{E}\left[\sum_{j=0}^{2N-2}\left|\eta_j \frac{1}{\sqrt{2N-1}}\right|^2\right] \\
=& \frac{1}{2N-1}\mathbb{E}[\|\eta\|_2^2].
\end{aligned}$$

Note that $f$ has degree at most $2d-1$, even when interpolated as a $2N-1$ degree trigonometric polynomial, so if the extra dimensions of $\tilde{f}_N$ are truncated, and we let

$$\tilde{f} = \sum_{j=0}^{2N-2} (f(\omega_{2N-1}^j) + \eta_j)\frac{1}{2N-1}D_{\omega_{2N-1}^j, d-1}$$

then, because each component of $\tilde{f}_N - f$ is independent, we get that

$$\mathbb{E}[\|\tilde{f} - f\|_2^2] = \frac{2d-1}{2N-1}\mathbb{E}[\|\tilde{f}_N - f\|_2^2] = \frac{2d-1}{(2N-1)^2}\mathbb{E}[\|\eta\|_2^2]$$

□

## 2.6 Phase retrieval algorithm: stability and oversampling

We begin proving that the phase retrieval algorithm presented in section 2.3 is stable by giving a result that assumes we are already given the approximating trigonometric polynomials that arise in step 1. In other words, we show that if given a fixed polynomial $p \in \mathcal{P}_d$ and approximating trigonometric polynomials $f_0 \approx |p|^2$, $f_1 \approx |(I - R_{0,\omega_d})p|^2$, and $f_2 \approx |(I - iR_{0,\omega_d})p|^2$ with sample values that are bounded away from zero, the recovery error of applying steps 2 and 3 is $O(\|f_0 - |p|^2\|_2)$.

**Lemma 2.6.1.** *Let $m \in (0,1]^d$. For any nonzero $p \in \mathcal{P}_d$, and any $f_0, f_1, f_2 \in \mathcal{R}_d$ such that $f_0 \approx |p|^2$, $f_1 \approx |(I - R_{0,\omega_d})p|^2$, and $f_2 \approx |(I - iR_{0,\omega_d})p|^2$, define $E \in \mathbb{R}^3_+$ such that $E = \{\|f_0 - |p|^2\|_2, \|f_1 - |(I - R_{0,\omega_d})p|^2\|_2, \|f_2 - |(I - iR_{0,\omega_d})p|^2\|_2\}$ and define $C(t_1, t_2) \in \mathbb{R}$ such that $C(t_1, t_2) = \frac{(1+\sqrt{2})\sqrt{2d-1}t_1 + d\|p\|_2^2}{t_2\sqrt{d}\|p\|_2^2}$. Then define $w$ as in lemma 2.4.3. If there exists a $z_0 \in \mathbb{T}$ such that $|p(z_0)|^2 \geq dm_0\|p\|_2^2$ and for all $j$ from $0$ to $d-1$, $|f_0(z_0\omega_d^j)| \geq dm_j\|p\|_2^2$, then an approximation $\tilde{p} \in \mathcal{P}_d$ can be obtained such that for some $c_0 \in \mathbb{T}$*

$$\|\tilde{p} - c_0 p\|_2 \leq \|w(C(\|E\|_\infty, \min(m)), \min(m))\|_2 \frac{\sqrt{2d}\|E\|_2}{\|p\|_2}$$

*by using steps 3 of the algorithm given in section 2.3 with the values $\{f_0(z_0\omega_d^j)\}_{j=0}^{d-1}$, $\{f_1(z_0\omega_d^j)\}_{j=0}^{d-2}$, and $\{f_2(z_0\omega_d^j)\}_{j=0}^{d-2}$.*

*Proof.* Recall that the set $\{\frac{1}{\sqrt{d}}K_{z_0\omega_d^j}\}_{j=0}^{d-1}$ is an ordered orthonormal basis for $\mathcal{P}_d$. Let $x$ be the vector $p$ represented in this ordered basis. Then for any $j$ from 0 to $d-1$,

$$|x_j|^2 = \left|\left\langle p, \tfrac{1}{\sqrt{d}} K_{z_0\omega_d^j}\right\rangle\right|^2 = \tfrac{1}{d}|\langle p, K_{z_0\omega_d^j}\rangle|^2 = \tfrac{1}{d}|p(z_0\omega_d^j)|^2 = \tfrac{1}{d}|p|^2(z_0\omega_d^j) = \tfrac{1}{d}\langle |p|^2, D_{z_0\omega_d^j, d-1}\rangle\,.$$

If we define $\epsilon \in \mathbb{R}^{3d-2}$ such that

$$\epsilon_j = \begin{cases} \tfrac{1}{d}\left(f_0(z_0\omega_d^j) - |p(z_0\omega_d^j)|^2\right) & \text{if } 0 \le j \le d-1 \\ \tfrac{1}{d}\left(f_1(z_0\omega_d^j) - |p(z_0\omega_d^j) - p(z_0\omega_d^{j+1})|^2\right) & \text{if } d \le j \le 2d-2 \\ \tfrac{1}{d}\left(f_2(z_0\omega_d^{j+1}) - |p(z_0\omega_d^{j+1}) - ip(z_0\omega_d^{j+2})|^2\right) & \text{if } 2d-1 \le j \le 3d-3 \end{cases}$$

then by Cauchy-Schwarz $\|\epsilon\|_\infty \le \frac{\sqrt{2d-1}}{d}\|E\|_\infty$ and by Lemma 2.5.1 $\|\epsilon\|_2 \le \sqrt{\frac{2}{d}}\|E\|_2$. Additionally, the values $\{f_0(z_0\omega_d^j)\}_{j=0}^{d-1}$, $\{f_1(z_0\omega_d^j)\}_{j=0}^{d-2}$, and $\{f_2(z_0\omega_d^j)\}_{j=0}^{d-2}$ are precisely equal to the values

$$\begin{aligned} \{|x_k|^2 + \epsilon_k\}_{k=0}^{d-1} \\ \{|x_k - x_{k+1}|^2 + \epsilon_{k+d}\}_{k=0}^{d-2} \\ \{|x_k - ix_{k+1}|^2 + \epsilon_{k+2d-1}\}_{k=0}^{d-2}\,. \end{aligned}$$

If Lemma 2.4.3 is applied to these measurements, and we use the equivalence of norms,

$$\|x\|_\infty = \max_k\left\{\left|\left\langle p, \frac{1}{\sqrt{d}}K_{z_0\omega_d^k}\right\rangle\right|\right\} \le \max_k\left\{\|p\|_2\left\|\frac{1}{\sqrt{d}}K_{z_0\omega_d^k}\right\|_2\right\} = \|p\|_2$$

and

$$\|x\|_\infty = \sqrt{\max_k\left\{\left|\left\langle p, \frac{1}{\sqrt{d}}K_{z_0\omega_d^k}\right\rangle\right|^2\right\}} \ge \sqrt{\frac{1}{d}\sum_k\left|\left\langle p, \frac{1}{\sqrt{d}}K_{z_0\omega_d^k}\right\rangle\right|^2} = \frac{1}{\sqrt{d}}\|p\|_2\,.$$

Then

$$C = \frac{(1+\sqrt{2})\|\epsilon\|_\infty + \|x\|_\infty^2}{\min(m)\|x\|_\infty^2} \leq \frac{(1+\sqrt{2})\frac{\sqrt{2d-1}}{d}\|E\|_\infty + \|p\|_2^2}{\min(m)\|p\|_2^2}\sqrt{d} = C(\|E\|_\infty, \min(m))\,.$$

We obtain a vector of coefficients $y \in \mathbb{C}^d$ by Lemma 2.4.3 that satisfies

$$\sum_{k=1}^{d}\left|y_k - \frac{\overline{p(\tilde{z}_0\omega_d)}}{|p(\tilde{z}_0\omega_d)|}p(\tilde{z}_0\omega_d^k)\right| \leq \|w(C, \min(m))\|_2 \frac{\|\epsilon\|_2}{\|x\|_\infty}$$

and because each component of $w$ is non-decreasing in $C$, and by the definition of $\epsilon$

$$\sum_{k=1}^{d}\left|y_k - \frac{\overline{p(\tilde{z}_0\omega_d)}}{|p(\tilde{z}_0\omega_d)|}p(\tilde{z}_0\omega_d^k)\right| \leq \|w(C(\|E\|_\infty, \min(m)), \min(m))\|_2 \frac{\sqrt{2}\|E\|_2}{\|p\|_2}\,.$$

Let $\tilde{p} = \sum_{k=0}^{d-1} y_k \frac{1}{\sqrt{d}} K_{\tilde{z}_0\omega_d^k}$. Then

$$\begin{aligned}
\left\|\tilde{p} - \frac{\overline{p(\tilde{z}_0\omega_d)}}{|p(\tilde{z}_0\omega_d)|}p\right\|_2 &= \left\|\sum_{k=0}^{d-1} y_k \frac{1}{\sqrt{d}} K_{\tilde{z}_0\omega_d^k} - \frac{\overline{p(\tilde{z}_0\omega_d)}}{|p(\tilde{z}_0\omega_d)|}\sum_{k=0}^{d-1} p(\tilde{z}_0\omega_d^k)\frac{1}{\sqrt{d}}K_{\tilde{z}_0\omega_d^k}\right\|_2 \\
&= \left\|\sum_{k=0}^{d-1}\left(y_k - \frac{\overline{p(\tilde{z}_0\omega_d)}}{|p(\tilde{z}_0\omega_d)|}p(\tilde{z}_0\omega_d^k)\right)\frac{1}{\sqrt{d}}K_{\tilde{z}_0\omega_d^k}\right\|_2 \\
&\leq \sum_{k=0}^{d-1}\left|y_k - \frac{\overline{p(\tilde{z}_0\omega_d)}}{|p(\tilde{z}_0\omega_d)|}p(\tilde{z}_0\omega_d^k)\right|\left\|\frac{1}{\sqrt{d}}K_{\tilde{z}_0\omega_d^k}\right\|_2 \\
&\leq \|w(C(\|E\|_\infty, \min(m)), \min(m))\|_2 \frac{\sqrt{2}\|E\|_2}{\|p\|_2}\,.
\end{aligned}$$

$\square$

With the above lemma, we may now give a stability result for the entire algorithm. Note that this result considers the effect of oversampling. If we let $N = d$, then we get the result with no oversampling, and if we let all $\eta = 0$, then we show the injectivity of our measurements.

**Theorem 2.6.2.** *Let $\tilde{m} \in (0,1]$. For any nonzero polynomial $p \in \mathcal{P}_d$, and any noise vectors $\eta_0, \eta_1, \eta_2 \in \mathbb{R}^{2N-1}$, if there exists $z_0 \in \mathbb{T}$ such that $|p(z_0\omega_d)|^2 \geq \tilde{m}d\|p\|_2^2$ and $\min\{|p(\omega_d^j z_0)|^2 - \frac{\sqrt{2d-1}}{\sqrt{2N-1}}\|\eta_0\|_2\}_{j=0}^{d-1} \geq \tilde{m}d\|p\|_2^2$, then an approximation $\tilde{p} \in \mathcal{P}_d$ can be obtained by the algorithm given in section 2.3 with the phaseless measurements*

$$\{|p|^2(\omega_{2N-1}^k) + (\eta_0)_k\}_{k=0}^{2N-2}$$

$$\{|(I - R_{0,\omega_d})p|^2(\omega_{2N-1}^k) + (\eta_1)_k\}_{k=0}^{2N-2}$$

$$\{|(I - iR_{0,\omega_d})p|^2(\omega_{2N-1}^k) + (\eta_2)_k\}_{k=0}^{2N-2}$$

*such that if $C(t_1, t_2) = \frac{(1+\sqrt{2})\sqrt{2d-1}t_1 + d\|p\|_2^2}{t_2\sqrt{d}\|p\|_2^2}$ then for some $c_0 \in \mathbb{T}$*

$$\|\tilde{p} - c_0 p\|_2 \leq \left\| w\left(C\left(\tfrac{1}{\sqrt{2N-1}}\max_{l=0,1,2}\{\|\eta_l\|_2\}, \tilde{m}\right), \tilde{m}\right)\right\|_2 \frac{\sqrt{2}}{\sqrt{2N-1}} \frac{\sqrt{\|\eta_0\|_2^2 + \|\eta_1\|_2^2 + \|\eta_2\|_2^2}}{\|p\|_2}.$$

*Proof.* Let $m \in (0,1]^d$ such that each entry $m_j = \tilde{m}$. Note any trigonometric polynomial in $\mathcal{T}_d$ may be recovered using Dirichlet kernel interpolation on $\mathcal{T}_N$ for any $N \geq d$, as shown in Lemma 2.5.2. This requires the use of point evaluations at $\{\omega_{2N-1}^j\}_{j=0}^{2N-2}$. Note that $|p|^2$, $|(I - R_{0,\omega_d})p|^2$, and $|(I - iR_{0,\omega_d})p|^2$ are in $\mathcal{T}_d$, and using the above , these functions may be interpolated from the values at $\{\omega_{2N-1}^j\}_{j=0}^{2N-2}$. Approximating trigonometric polynomials $f_0 \approx |p|^2$, $f_1 \approx |(I - R_{0,\omega_d})p|^2$, and $f_2 \approx |(I - iR_{0,\omega_d})p|^2$ are obtained from this interpolation. Then by Lemma 2.5.2

$$\|f_0 - |p|^2\|_2 \leq \frac{1}{\sqrt{2N-1}}\|\eta_0\|_2\,,$$

$$\|f_1 - |(I - R_{0,\omega_d})p|^2\|_2 \leq \frac{1}{\sqrt{2N-1}}\|\eta_1\|_2\,,$$

and

$$\|f_2 - |(I - iR_{0,\omega_d})p|^2\|_2 \leq \frac{1}{\sqrt{2N-1}}\|\eta_2\|_2\,.$$

Note that for any $z \in \mathbb{T}$,

$$\begin{aligned}
|f_0(z)| =& \left| |p(z)|^2 + \sum_{j=0}^{2N-2} \eta_{0,j} \frac{1}{2N-1} D_{\omega_{2N-1}^j, d-1}(z) \right| \\
\geq& |p(z)|^2 - \left| \sum_{j=0}^{2N-2} \eta_{0,j} \frac{1}{2N-1} D_{\omega_{2N-1}^j, d-1}(z) \right| \\
=& |p(z)|^2 - \left| \left\langle \sum_{j=0}^{2N-2} \eta_{0,j} \frac{1}{\sqrt{2N-1}} D_{\omega_{2N-1}^j, d-1}, \frac{1}{\sqrt{2N-1}} D_{z,d-1} \right\rangle \right| \\
\geq& |p(z)|^2 - \left\| \frac{1}{\sqrt{2N-1}} D_{z,d-1} \right\|_2 \left\| \sum_{j=0}^{2N-2} \eta_{0,j} \frac{1}{\sqrt{2N-1}} D_{\omega_{2N-1}^j, d-1} \right\|_2 \\
\geq& |p(z)|^2 - \left\| \frac{1}{\sqrt{2N-1}} D_{z,d-1} \right\|_2 \left\| \sum_{j=0}^{2N-2} \eta_{0,j} \frac{1}{\sqrt{2N-1}} D_{\omega_{2N-1}^j, N-1} \right\|_2 \\
\geq& |p(z)|^2 - \frac{\sqrt{2d-1}}{\sqrt{2N-1}} \|\eta_0\|_2 \\
\geq& \tilde{m} d \|p\|_2^2 .
\end{aligned}$$

Thus, we may apply lemma 2.6.1 to get an approximation $\tilde{p} \in \mathcal{P}_d$ such that for some $c_0 \in \mathbb{T}$

$$\|\tilde{p} - c_0 p\|_2 \leq \|w(C(\|E\|_\infty, \min(m)), m)\|_2 \frac{\sqrt{2}\|E\|_2}{\|p\|_2} .$$

where $E = \{\|f_0 - |p|^2\|_2, \|f_1 - |(I - R_{0,\omega_d})p|^2\|_2, \|f_2 - |(I - iR_{0,\omega_d})p|^2\|_2\}$. Note that $\|E\|_\infty \leq \frac{1}{\sqrt{2N-1}} \max_{l=0,1,2}\{\|\eta_l\|_2\}$ and $\|E\|_2 \leq \frac{1}{\sqrt{2N-1}} \sqrt{\|\eta_0\|_2^2 + \|\eta_1\|_2^2 + \|\eta_2\|_2^2}$. Thus $C(\|E\|_\infty, \min(m)) \leq C(\frac{1}{\sqrt{2N-1}} \max_{l=0,1,2}\{\|\eta_l\|_2\}, \tilde{m})$. Then because all components of $w$ are increasing in $C$,

$$\|\tilde{p} - c_0 p\|_2 \leq \left\| w\left( C\left( \frac{1}{\sqrt{2N-1}} \max_{l=0,1,2}\{\|\eta_l\|_2\}, \tilde{m} \right), \tilde{m} \right) \right\|_2 \frac{\sqrt{2}}{\sqrt{2N-1}} \frac{\sqrt{\|\eta_0\|_2^2 + \|\eta_1\|_2^2 + \|\eta_2\|_2^2}}{\|p\|_2} .$$

$\square$

To obtain a uniform error bound that only assumes bounds on the norms of the vector $p$ and on the magnitude of the noise $\|\eta_0\|_2$, we use the max-min principle from appendix A.1. This provides us with a universally valid lower bound $\tilde{m}$ that applies to the above theorem. The oversampling results from the previous theorem are maintained.

**Theorem 2.6.3.** *Let $r = \sin(\frac{2\pi}{(d-1)d^2})$, and $0 < \alpha < 1$. For any nonzero $p \in \mathcal{P}_d$, and any noise vectors $\eta_0, \eta_1, \eta_2 \in \mathbb{R}^{2N-1}$, if $\beta = \frac{r^{\frac{(d-1)d}{2}}\left(\frac{d-1}{2d}\right)^d \frac{2}{d-1}}{\left(\prod_{k=1}^{d-1}(r^k+1)\right)}$ and $\|\eta_0\|_2 \leq \frac{\sqrt{2N-1}}{\sqrt{2d-1}}\alpha\beta^2\|p\|_2^2$, then an approximation $\tilde{p} \in \mathcal{P}_d$ can be obtained by the algorithm given in section 2.3 with the phaseless measurements*

$$\{|p|^2(\omega_{2N-1}^k) + (\eta_0)_k\}_{k=0}^{2N-2}$$
$$\{|(I - R_{0,\omega_d})p|^2(\omega_{2N-1}^k) + (\eta_1)_k\}_{k=0}^{2N-2}$$
$$\{|(I - iR_{0,\omega_d})p|^2(\omega_{2N-1}^k) + (\eta_2)_k\}_{k=0}^{2N-2}$$

*such that if $C = \frac{(1+\sqrt{2})\frac{\sqrt{2d-1}}{\sqrt{2N-1}}\max_{l=0,1,2}\{\|\eta_l\|_2\}+d\|p\|_2^2}{\beta^2(1-\alpha)\|p\|_2^2}\sqrt{d}$ then for some $c_0 \in \mathbb{T}$*

$$\|\tilde{p} - c_0 p\|_2 \leq \left\| w\left(C, \frac{1}{d}\beta^2(1-\alpha)\right)\right\|_2 \frac{\sqrt{2}}{\sqrt{2N-1}} \frac{\sqrt{\|\eta_0\|_2^2 + \|\eta_1\|_2^2 + \|\eta_2\|_2^2}}{\|p\|_2}.$$

*Proof.* By Lemma A.1.3 we know that there exists a $z_0 \in \mathbb{T}$ such that the distance between any element of $\{\omega_d^j z_0\}_{j=0}^{d-1}$ and any roots of any nonzero truncations of $p$ is at least $r$. Then by Lemma A.1.4 we know that for all $j$ from 0 to $d-1$, $|p(\omega_d^j z_0)| \geq \beta\|p\|_1 \geq \beta\|p\|_2$ where we define $\|p\|_1$ to be the $\ell_1$ norm of the monomial coefficients of $p$. Thus, we know that there exists a $z_0 \in \mathbb{T}$ such that

$$\min\left\{|p(\omega_d^j z_0)|^2 - \frac{\sqrt{2d-1}}{\sqrt{2N-1}}\|\eta_0\|_1\right\}_{j=0}^{d-1} \geq \beta^2\|p\|_2^2 - \frac{\sqrt{2d-1}}{\sqrt{2N-1}}\|\eta_0\|_1 \geq \beta^2(1-\alpha)\|p\|_2^2$$

for all $j$ from 0 to $d-1$ and we may use $z_0$ and $\tilde{m} = \frac{1}{d}\beta^2(1-\alpha)$ in the preceding theorem. When we apply the above theorem, we get

$$\|\tilde{p} - c_0 p\|_2 \leq \left\| w\left(C, \frac{1}{d}\beta^2(1-\alpha)\right)\right\|_2 \frac{\sqrt{2}}{\sqrt{2N-1}} \frac{\sqrt{\|\eta_0\|_2^2 + \|\eta_1\|_2^2 + \|\eta_2\|_2^2}}{\|p\|_2}.$$

□

We remark that many of the inequalities used to create this bound are not sharp. Thus the above theorem would benefit from an improved lower bound on the minimum magnitude. In fact, the value $\left\|w\left(C, \frac{1}{d}\beta^2(1-\alpha)\right)\right\|_2$ grows extremely fast with the dimension. Errors as large as would be indicated by this value have not appeared, even for the worst-case polynomial that has been found experimentally.

In figure 2.1, we show the results of oversampling in our algorithm, when the signal is the experimentally found worst-case, and the input noise with fixed $\ell_\infty$ norm equal to $10^{-9}$ has been chosen to take advantage of this polynomial. When $N = d = 7$, the $\ell_2$ error is $10^6$ times the $\ell_\infty$ norm of the input noise. As the oversampling rate increases, the recovery error drops, as predicted by the theorem.

### 2.6.1 A lower bound for the error resulting from a pathological signal

The above error inequalities are much worse than even the worst-case input error for the worst-case polynomial. But we don't have an analytic formula for the worst-case polynomial or its stability. Experimentally, the polynomial that is most affected by noise that has been found has roots whose angles are very close to the Chebyshev nodes. That is, if we found the $d-1$ Chebyshev nodes on the interval $[-\frac{\pi}{d}, \frac{\pi}{d}]$, and then created the polynomial with roots all on the unit circle whose angles occur at the Chebyshev nodes, then this new polynomial would be very close to our experimentally found worst-case

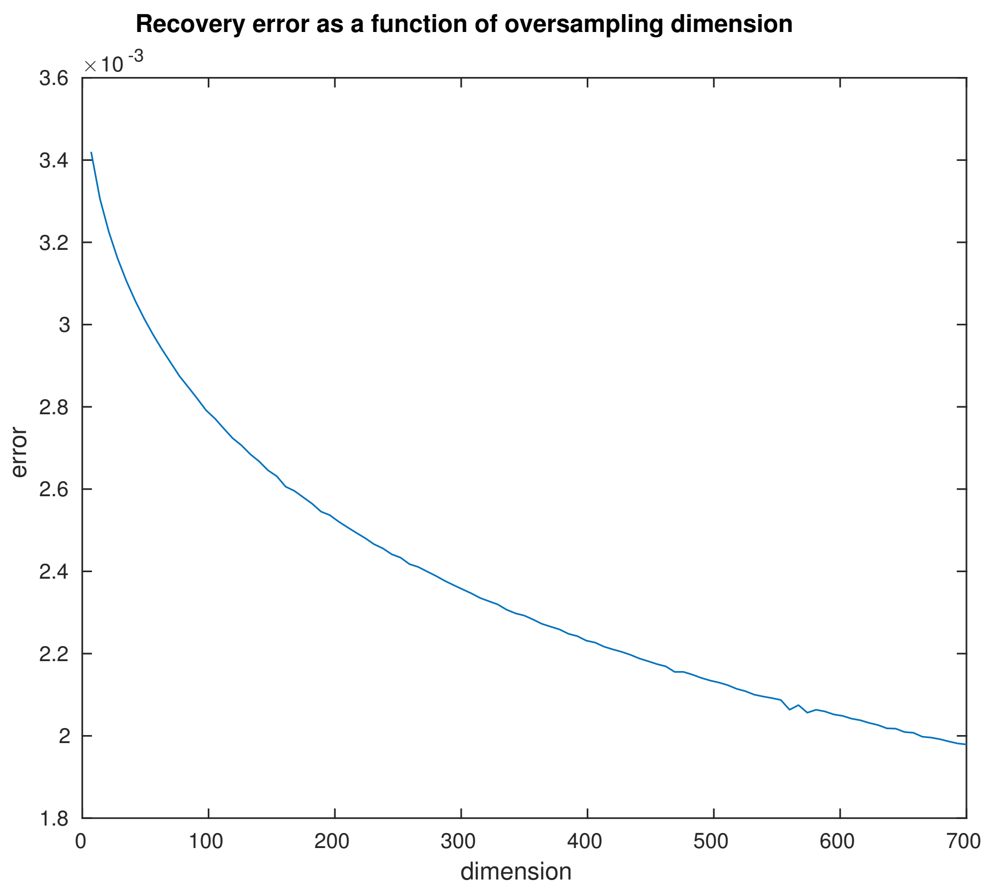


Figure 2.1: The $L^2$ distance between the signal recovered by the phase retrieval algorithm and the actual signal for a fixed polynomial in $d = 7$, with input noise with $\|\eta_0\|_\infty = 10^{-9}$ as a function of the oversampling dimension. The fixed polynomial that was chosen is the polynomial that most amplifies the noise.

polynomial. In this section, we shall provide a lower bound on the maximum error that can be achieved by using the algorithm given in section 2.3.

**Proposition 2.6.4.** *Let $p \in \mathcal{P}_d$ be a polynomial with roots*

$$z_k = e^{\frac{i\pi}{d}\cos\left(\frac{(2k-1)\pi}{2(d-1)}\right)}.$$

*Then $p(z) = a\prod_{k=1}^{d-1}(z - z_k)$ for some $a \in \mathbb{C}$ and*

$$\begin{aligned}\max\left\{|p(z)| : |z| = 1 \text{ and } \arg(z) \in [-\tfrac{\pi}{d}, \tfrac{\pi}{d}]\right\} \geq & \left|p\left(e^{\frac{i\pi}{d}}\right)\right| \\ =&|a|\prod_{k=1}^{d-1}\left|e^{\frac{i\pi}{d}\cos\left(\frac{(2k-1)\pi}{2(d-1)}\right)} - e^{\frac{i\pi}{d}}\right| \\ =&|a|\prod_{k=1}^{d-1}\left|e^{\frac{i\pi}{d}\left(\cos\left(\frac{(2k-1)\pi}{2(d-1)}\right)-1\right)} - 1\right|.\end{aligned}$$

Now we show the location of the maximum does not change when we map an appropriate polynomial on a line segment to a polynomial on an arc of the circle.

**Lemma 2.6.5.** *Let $d$ be an even integer with $d \geq 2$, let $\alpha \in (0, \frac{\pi}{2}]$, and let $\{a_k\}_{k=-\frac{d}{2},\, k\neq 0}^{\frac{d}{2}}$ be a set of numbers in the interval $[-\alpha, \alpha]$ such that $a_{-k} = -a_k$. Let $p_0 \in \mathcal{P}_d$ be the real polynomial such that*

$$p_0(x) = \prod_{\substack{k=-\frac{d}{2} \\ k\neq 0}}^{\frac{d}{2}}(x - a_k) = \prod_{k=1}^{\frac{d}{2}}(x - a_{-k})(x - a_k)$$

*and let $p_1 \in \mathcal{P}_d$ be the polynomial such that*

$$p_1(z) = \prod_{\substack{k=-\frac{d}{2} \\ k\neq 0}}^{\frac{d}{2}}(z - e^{ia_k}) = \prod_{k=1}^{\frac{d}{2}}(z - e^{ia_{-k}})(z - e^{ia_k}).$$

*If $p_0$ achieves its maximum over the interval $[-\alpha, \alpha]$ at $x = 0$, then $p_1$ achieves its maximum over the set $\{z : |z| = 1$ and $\arg(z) \in [-\alpha, \alpha]\}$ uniquely at $z = 1$.*

*Proof.* Note that for all $z \in \{z : |z| = 1 \text{ and } \arg(z) \in [-\alpha, \alpha]\}$ with $\arg(z) \notin \{a_k\}_{k=-\frac{d}{2},\, k\neq 0}^{\frac{d}{2}}$

$$\begin{aligned}
|p_1(z)| &= \prod_{\substack{k=-\frac{d}{2} \\ k\neq 0}}^{\frac{d}{2}} |z - e^{ia_k}| \\
&= \prod_{\substack{k=-\frac{d}{2} \\ k\neq 0}}^{\frac{d}{2}} \frac{|z - e^{ia_k}|}{|\arg(z) - a_k|} |\arg(z) - a_k| \\
&= |p_0(\arg(z))| \prod_{\substack{k=-\frac{d}{2} \\ k\neq 0}}^{\frac{d}{2}} \frac{|z - e^{ia_k}|}{|\arg(z) - a_k|} \\
&= |p_0(\arg(z))| \prod_{\substack{k=-\frac{d}{2} \\ k\neq 0}}^{\frac{d}{2}} \operatorname{sinc}\left(\frac{|\arg(z) - a_k|}{2}\right) \\
&= |p_0(\arg(z))| \prod_{k=1}^{\frac{d}{2}} \operatorname{sinc}\left(\frac{|\arg(z) - a_{-k}|}{2}\right) \operatorname{sinc}\left(\frac{|\arg(z) - a_k|}{2}\right)
\end{aligned}$$

Note that because $\alpha \leq \frac{\pi}{2}$, we know that $|\arg(z) - a_k| \leq 2\alpha \leq \pi$ and so we are only looking at the sinc function on the interval $(0, \frac{\pi}{2}]$. On this interval, sinc is strictly positive, strictly decreasing, and strictly concave. Because it is strictly positive, we may use the inequality of arithmetic and geometric means to show that for any $k$,

$$\operatorname{sinc}\left(\frac{|\arg(z) - a_{-k}|}{2}\right) \operatorname{sinc}\left(\frac{|\arg(z) - a_k|}{2}\right) \leq \left(\frac{\operatorname{sinc}\left(\frac{|\arg(z) - a_{-k}|}{2}\right) + \operatorname{sinc}\left(\frac{|\arg(z) - a_k|}{2}\right)}{2}\right)^2$$

and equality holds if and only if

$$\operatorname{sinc}\left(\frac{|\arg(z)-a_{-k}|}{2}\right)=\operatorname{sinc}\left(\frac{|\arg(z)-a_{k}|}{2}\right)$$

which is true if and only if $|\arg(z)-a_{-k}| = |\arg(z)-a_k|$, which is true if and only if $\arg(z)=0$. By the strict concavity of sinc, we know that for any $k$,

$$\frac{\operatorname{sinc}\left(\frac{|\arg(z)-a_{-k}|}{2}\right)+\operatorname{sinc}\left(\frac{|\arg(z)-a_{k}|}{2}\right)}{2}\leq \operatorname{sinc}\left(\frac{\frac{|\arg(z)-a_{-k}|}{2}+\frac{|\arg(z)-a_{k}|}{2}}{2}\right)$$

and equality holds if and only if $|\arg(z)-a_{-k}| = |\arg(z)-a_k|$, which is true if and only if $\arg(z)=0$. Thus,

$$|p_1(z)|\leq |p_0(\arg(z))|\prod_{k=1}^{\frac{d}{2}}\operatorname{sinc}^2\left(\frac{|\arg(z)-a_{-k}|+|\arg(z)-a_k|}{4}\right)$$

and equality holds if and only if $z=1$. Note that for any $k$, if $|\arg(z)|>|a_k|$, then

$$|\arg(z)-a_{-k}|+|\arg(z)-a_k| = |\arg(z)-a_{-k}+\arg(z)-a_k| = 2|\arg(z)|>2|a_k|$$

and if $-|a_k|\leq \arg(z)\leq |a_k|$, then

$$|\arg(z)-a_{-k}|+|\arg(z)-a_k| = |a_k|-\arg(z)+\arg(z)+|a_k| = 2|a_k|\,.$$

Thus, for all $k$, $|\arg(z)-a_{-k}|+|\arg(z)-a_k|\geq 2|a_k|$, and because sinc is strictly decreasing, we get that

$$\operatorname{sinc}^2\left(\frac{|\arg(z)-a_{-k}|+|\arg(z)-a_k|}{4}\right)\leq \operatorname{sinc}^2\left(\frac{|a_k|}{2}\right)\,.$$

Thus,

$$|p_1(z)| \leq |p_0(\arg(z))| \prod_{k=1}^{\frac{d}{2}} \operatorname{sinc}^2\left(\frac{|a_k|}{2}\right)$$

and equality holds if and only if $z = 1$. Because $|p_0(\arg(z))|$ achieves its maximum at $\arg(z) = 0$, we know that $|p_1(z)|$ achieves its maximum at $z = 1$, and we know that it doesn't achieve this maximum anywhere else on the set we are concerned with, because the above inequality does not hold for $z \neq 1$. □

We also want a lower bound for the values of the pathological polynomial on the arc with angles in $[-\frac{\pi}{d}, \frac{\pi}{d}]$.

**Proposition 2.6.6.** *Let $d \geq 3$ and let $p \in \mathcal{P}_d$ be a polynomial with roots*

$$z_k = e^{\frac{i\pi}{d}\cos\left(\frac{(2k-1)\pi}{2(d-1)}\right)}.$$

*Then $p(z) = a\prod_{k=1}^{d-1}(z - z_k)$ for some $a \in \mathbb{C}$. For any $z$ with $|z| = 1$ and angle inside the interval $[-\frac{\pi}{d}, \frac{\pi}{d}]$,*

$$|p(z)| \leq |a| \left(\frac{\pi}{d}\right)^{d-1}.$$

*Proof.* Let $p_0$ be the polynomial that is equal to $\frac{\left|e^{\frac{i\pi}{d}}-1\right|}{z-1}p(z)$ if $p(z)$ has a root at 1, and equal to $p(z)$ otherwise. Then on this interval, $|z-1| \leq \left|e^{\frac{i\pi}{d}} - 1\right|$, so $|p(z)| \leq |p_0(z)|$. By the previous lemma, we know that $|p_0(z)| \leq |p_0(1)|$. Thus,

$$\begin{aligned}
|p(z)| &\leq |p_0(z)| \\
&\leq |p_0\left(1\right)| \\
&= |a| \left| e^{\frac{i\pi}{d}} - 1 \right|^{2\left(\frac{d-1}{2} - \lfloor \frac{d-1}{2} \rfloor\right)} \prod_{\substack{k=1 \\ k \neq \frac{d}{2}}}^{d-1} \left| e^{\frac{i\pi}{d} \cos\left(\frac{(2k-1)\pi}{2(d-1)}\right)} - 1 \right| \\
&= |a| \left| e^{\frac{i\pi}{d}} - 1 \right|^{2\left(\frac{d-1}{2} - \lfloor \frac{d-1}{2} \rfloor\right)} \prod_{k=1}^{\lfloor \frac{d-1}{2} \rfloor} \left| e^{\frac{i\pi}{d} \cos\left(\frac{(2k-1)\pi}{2(d-1)}\right)} - 1 \right|^2
\end{aligned}$$

and because distance along the arc is greater than linear distance

$$\begin{aligned}
&\leq |a| \left| \frac{\pi}{d} \right|^{2\left(\frac{d-1}{2} - \lfloor \frac{d-1}{2} \rfloor\right)} \prod_{k=1}^{\lfloor \frac{d-1}{2} \rfloor} \left| \frac{\pi}{d} \cos\left(\frac{(2k-1)\pi}{2(d-1)}\right) \right|^2 \\
&= |a| \left( \frac{\pi}{d} \right)^{2\left(\frac{d-1}{2} - \lfloor \frac{d-1}{2} \rfloor\right)} \left( \frac{\pi}{d} \right)^{2\lfloor \frac{d-1}{2} \rfloor} \prod_{k=1}^{\lfloor \frac{d-1}{2} \rfloor} \cos^2\left(\frac{(2k-1)\pi}{2(d-1)}\right) \\
&= |a| \left( \frac{\pi}{d} \right)^{d-1} \prod_{k=1}^{\lfloor \frac{d-1}{2} \rfloor} \cos^2\left(\frac{(2k-1)\pi}{2(d-1)}\right) \\
&\leq |a| \left( \frac{\pi}{d} \right)^{d-1}.
\end{aligned}$$

□

We will also get a lower bound for the value of the pathological polynomial at $\omega_d$, which is the second coordinate in the ordered basis of point evaluations.

**Proposition 2.6.7.** *Let $d \geq 6$ and let $p \in \mathcal{P}_d$ be a polynomial with roots*

$$z_k = e^{\frac{i\pi}{d} \cos\left(\frac{(2k-1)\pi}{2(d-1)}\right)}.$$

*Then* $p(z) = a\prod_{k=1}^{d-1}(z - z_k)$ *for some* $a \in \mathbb{C}$*, and*

$$|p(e^{\frac{2i\pi}{d}})| \geq |a| 2^{\frac{d-1}{2}} \left(\frac{\pi}{d}\right)^{d-1} \frac{3\sqrt{3}(5+\sqrt{5})}{64}.$$

*Proof.*

$$\begin{aligned}
|p(e^{\frac{2i\pi}{d}})| &= |a| \prod_{k=1}^{d-1} \left| e^{\frac{2i\pi}{d}} - e^{\frac{i\pi}{d}\cos\left(\frac{(2k-1)\pi}{2(d-1)}\right)} \right| \\
&\geq |a| \prod_{k=1}^{\lfloor\frac{d-1}{2}\rfloor} \left| e^{\frac{2i\pi}{d}} - e^{\frac{i\pi}{d}} \right| \prod_{k=\lfloor\frac{d-1}{2}\rfloor+1}^{d-1} \left| e^{\frac{2i\pi}{d}} - 1 \right| \\
&= |a| \left| e^{\frac{i\pi}{d}} - 1 \right|^{\lfloor\frac{d-1}{2}\rfloor} \left| e^{\frac{2i\pi}{d}} - 1 \right|^{d-1-\lfloor\frac{d-1}{2}\rfloor} \\
&= |a| \left| 2\sin\left(\frac{\pi}{2d}\right) \right|^{\lfloor\frac{d-1}{2}\rfloor} \left| 2\sin\left(\frac{\pi}{d}\right) \right|^{d-1-\lfloor\frac{d-1}{2}\rfloor}.
\end{aligned}$$

Note that, for any $x \in [0, \pi]$, $\sin(x) \geq x\cos(x)$. Thus,

$$\begin{aligned}
|p(e^{\frac{2i\pi}{d}})| &\geq |a| \left| 2\sin\left(\frac{\pi}{2d}\right) \right|^{\lfloor\frac{d-1}{2}\rfloor} \left| 2\sin\left(\frac{\pi}{d}\right) \right|^{d-1-\lfloor\frac{d-1}{2}\rfloor} \\
&\geq |a| 2^{d-1} \left| \frac{\pi}{2d}\cos\left(\frac{\pi}{2d}\right) \right|^{\lfloor\frac{d-1}{2}\rfloor} \left| \frac{\pi}{d}\cos\left(\frac{\pi}{d}\right) \right|^{d-1-\lfloor\frac{d-1}{2}\rfloor} \\
&= |a| 2^{d-1-\lfloor\frac{d-1}{2}\rfloor} \left(\frac{\pi}{d}\right) \left| \cos\left(\frac{\pi}{2d}\right) \right|^{\lfloor\frac{d-1}{2}\rfloor} \left| \cos\left(\frac{\pi}{d}\right)^{d-1} \right|^{d-1-\lfloor\frac{d-1}{2}\rfloor} \\
&\geq |a| 2^{d-1-\lfloor\frac{d-1}{2}\rfloor} \left(\frac{\pi}{d}\right)^{d-1} \left| \cos\left(\frac{\pi}{2d}\right) \right|^{\frac{d-1}{2}} \left| \cos\left(\frac{\pi}{d}\right) \right|^{\frac{d}{2}}.
\end{aligned}$$

Note that for $d \geq 3$, the derivative of $\left|\cos\left(\frac{\pi}{2d}\right)\right|^{\frac{d-1}{2}}$ is positive. Thus for $d \geq 5$, $\left|\cos\left(\frac{\pi}{2d}\right)\right|^{\frac{d-1}{2}}$ is increasing and $\left|\cos\left(\frac{\pi}{2d}\right)\right|^{\frac{d-1}{2}} \geq \left|\cos\left(\frac{\pi}{10}\right)\right|^{2} = \frac{5+\sqrt{5}}{8}$. Also note that for $d > 2$, the derivative of $\left|\cos\left(\frac{\pi}{d}\right)\right|^{\frac{d}{2}}$ is positive. Thus for $d \geq 6$, $\left|\cos\left(\frac{\pi}{d}\right)\right|^{\frac{d}{2}}$ is increasing and we get

$\left|\cos\left(\frac{\pi}{d}\right)\right|^{\frac{d}{2}} \geq \left|\cos\left(\frac{\pi}{6}\right)\right|^{3} = \frac{3\sqrt{3}}{8}$. Thus,

$$\begin{aligned}
|p(e^{\frac{2i\pi}{d}})| &\geq |a| 2^{d-1-\lfloor\frac{d-1}{2}\rfloor} \left(\frac{\pi}{d}\right)^{d-1} \left|\cos\left(\frac{\pi}{2d}\right)\right|^{\frac{d-1}{2}} \left|\cos\left(\frac{\pi}{d}\right)\right|^{\frac{d}{2}} \\
&\geq |a| 2^{d-1-\lfloor\frac{d-1}{2}\rfloor} \left(\frac{\pi}{d}\right)^{d-1} \frac{3\sqrt{3}(5+\sqrt{5})}{64} \\
&= |a| 2^{\frac{d-1}{2}} 2^{\frac{d-1}{2}-\lfloor\frac{d-1}{2}\rfloor} \left(\frac{\pi}{d}\right)^{d-1} \frac{3\sqrt{3}(5+\sqrt{5})}{64} \\
&\geq |a| 2^{\frac{d-1}{2}} \left(\frac{\pi}{d}\right)^{d-1} \frac{3\sqrt{3}(5+\sqrt{5})}{64}.
\end{aligned}$$

□

And finally, we establish a ratio between the first two coordinates of the pathological polynomial with respect to the point evaluation basis.

**Corollary 2.6.8.** *Let $d$ be an even integer with $d \geq 6$ and let $p \in \mathcal{P}_d$ be a polynomial with roots*

$$z_k = e^{\frac{i\pi}{d}\cos\left(\frac{(2k-1)\pi}{2(d-1)}\right)}.$$

*Then $p(z) = a\prod_{k=1}^{d-1}(z - z_k)$ for some $a \in \mathbb{C}$, and*

$$\frac{|p(1)|}{|p(e^{\frac{2i\pi}{d}})|} \leq \frac{|a|\left(\frac{\pi}{d}\right)^{d-1}}{|a| 2^{\frac{d-1}{2}} \left(\frac{\pi}{d}\right)^{d-1} \frac{3\sqrt{3}(5+\sqrt{5})}{64}} = \frac{64}{2^{\frac{d-1}{2}} 3\sqrt{3}(5+\sqrt{5})}$$

*and if $d \geq 11$, then*

$$\frac{|p(1)|}{|p(e^{\frac{2i\pi}{d}})|} \leq \frac{64}{2^{\frac{d-1}{2}} 3\sqrt{3}(5+\sqrt{5})} \leq \frac{64}{2^5 3\sqrt{3}(5+\sqrt{5})} = \frac{2}{3\sqrt{3}(5+\sqrt{5})} \leq \frac{(\sqrt{2}-1)^2}{2\sqrt{2}}$$

Just as we did with the main result, we prove a claim regarding step 3 of the algorithm for our pathological vector, before we show that this applies to the entire algorithm.

**Lemma 2.6.9.** *For any $x \in \mathbb{C}^d$, if $\epsilon \in \mathbb{R}^{3d-2}$ such that $\epsilon_0 \neq 0$ and all other components*

*$\epsilon_j = 0$ and if $\{e_j\}_{j=0}^{d-1}$ is an orthonormal basis such that $x$ is full with respect to $\{e_j\}_{j=0}^{d-1}$, and $|\langle x, e_0\rangle|^2 + \epsilon_0 > 0$, then a vector $y$ may be obtained such that for some $\xi$ between $|x_0|^2 + \epsilon_0$ and $|x_0|^2$,*

$$y_0 - \frac{\overline{x_0}}{|x_0|}x_0 = \frac{\epsilon_0}{2\sqrt{\xi}}$$

*and for all $k$ from 1 to $d-1$,*

$$y_k - \frac{\overline{x_0}}{|x_0|}x_k = \frac{\overline{x_1}x_k}{|x_1|^2}\left(\frac{\frac{1}{2}(1-i)\sqrt{|x_0|^2+\epsilon_0} + \overline{x_0}x_1\left(\frac{1}{2\sqrt{\xi}} - \frac{1}{|x_0|}\right)}{|x_0|^2+\epsilon_0}\right)\epsilon_0$$

*by using the generalized step 3 algorithm presented in section 2.4 with the phaseless measurements*

$$\{|\langle x, e_k\rangle|^2 + \epsilon_k\}_{k=0}^{d-1}$$
$$\{|\langle x, e_k - e_{k+1}\rangle|^2 + \epsilon_{k+d}\}_{k=0}^{d-2}$$
$$\{|\langle x, e_k - ie_{k+1}\rangle|^2 + \epsilon_{k+2d-1}\}_{k=0}^{d-2}.$$

*Proof.* For the case $k = 0$, we let $y_0 = \sqrt{|\langle x, e_0\rangle|^2 + \epsilon_0}$. Then by the mean value theorem and concavity of the square root, there exists a $\xi$ between $|\langle x, e_0\rangle|^2 + \epsilon_0$ and $|\langle x, e_0\rangle|^2$ such that

$$\begin{aligned}\left(y_0 - \frac{\overline{x_0}}{|x_0|}x_0\right) &= \sqrt{|\langle x, e_0\rangle|^2 + \epsilon_0} - \sqrt{|\langle x, e_0\rangle|^2} \\ &= \frac{\epsilon_0}{2\sqrt{\xi}}\,.\end{aligned}$$

For other values of $k$, the proof proceeds by induction on $k$. The base case for this induction is the case $k = 1$. Note that for all $k$ from 0 to $d-1$,

$$\overline{x_k}x_{k+1} = \tfrac{1}{2}\left((1-i)|x_k|^2 + (1-i)|x_{k+1}|^2 - |x_k - x_{k+1}|^2 + i|x_k - ix_{k+1}|^2\right)$$

so an approximation for $\overline{x_k}x_{k+1}$ may be obtained as a linear combination of the perturbed measurements. If we let

$$\begin{aligned} t_k =& \tfrac{1}{2}\left((1-i)\left(|\langle x, e_k\rangle|^2 + \epsilon_k + |\langle x, e_{k+1}\rangle|^2 + \epsilon_{k+1}\right)\right. \\ &\left. - \left(|\langle x, e_k - e_{k+1}\rangle|^2 + \epsilon_{k+d}\right) + i\left(|\langle x, e_k - ie_{k+1}\rangle|^2 + \epsilon_{k+2d-1}\right)\right) \end{aligned}$$

and

$$y_{k+1} = \frac{t_k}{|\langle x, e_k\rangle|^2 + \epsilon_k} y_k \, ,$$

then $t_k \approx \overline{x_k}x_{k+1}$ and $y_{k+1} \approx \frac{\overline{x_k}x_{k+1}}{|\langle x,e_k\rangle|^2+\epsilon_k} y_k$. A direct computation shows the error for the approximation of the term used in phase propagation,

$$\begin{aligned} t_k - \overline{x_k}x_{k+1} =& \left(\frac{(1-i)(|x_k|^2 + \epsilon_k) + (1-i)(|x_{k+1}|^2 + \epsilon_{k+1})}{2}\right. \\ &- \frac{(|x_k - x_{k+1}|^2 + \epsilon_{d+k}) - i(|x_k - ix_{k+1}|^2 + \epsilon_{2d+k-1})}{2} \\ &- \frac{(1-i)|x_k|^2 + (1-i)|x_{k+1}|^2}{2} \\ &\left. + \frac{|x_k - x_{k+1}|^2 - i|x_k - ix_{k+1}|^2}{2}\right) \\ =& \left(\frac{(1-i)\epsilon_k + (1-i)\epsilon_{k+1} - \epsilon_{d+k} + i\epsilon_{2d+k-1}}{2}\right) \end{aligned}$$

By the assumption on the components of $\epsilon$,

$$t_k - \overline{x_k}x_{k+1} = \delta_{0,k}\frac{(1-i)\epsilon_0}{2} \, .$$

We then directly compute the relationship between the vector and approximate recovery,

$$
\begin{aligned}
y_{k+1} - \frac{\overline{x_0}}{|x_0|}x_{k+1} =& \frac{t_k}{|\langle x, e_k\rangle|^2 + \epsilon_k} y_k - \frac{\overline{x_k}x_{k+1}}{|x_k|^2}\frac{\overline{x_0}}{|x_0|}x_k \\
=& \frac{|x_k|^2 t_k y_k - \overline{x_k}x_{k+1}\frac{\overline{x_0}}{|x_0|}x_k(|x_k|^2+\epsilon_k)}{(|x_k|^2+\epsilon_k)|x_k|^2} \\
=& \frac{|x_k|^2(t_k - \overline{x_k}x_{k+1})y_k + |x_k|^2\overline{x_k}x_{k+1}(y_k - \frac{\overline{x_0}}{|x_0|}x_k) - \overline{x_k}x_{k+1}\frac{\overline{x_0}}{|x_0|}x_k\epsilon_k}{(|x_k|^2+\epsilon_k)|x_k|^2} \\
=& \frac{(t_k - \overline{x_k}x_{k+1})y_k + \overline{x_k}x_{k+1}(y_k - \frac{\overline{x_0}}{|x_0|}x_k) - x_{k+1}\frac{\overline{x_0}}{|x_0|}\epsilon_k}{|x_k|^2+\epsilon_k}
\end{aligned}
$$

In the base ($k = 1$) case, we plug in $k = 0$ into the above equation to get

$$
\begin{aligned}
y_1 - \frac{\overline{x_0}}{|x_0|}x_1 =& \frac{(t_0 - \overline{x_0}x_1)y_0 + \overline{x_0}x_1(y_0 - \frac{\overline{x_0}}{|x_0|}x_0) - x_1\frac{\overline{x_0}}{|x_0|}\epsilon_0}{|x_0|^2+\epsilon_0} \\
=& \frac{\frac{(1-i)\epsilon_0}{2}y_0 + \overline{x_0}x_1(y_0 - \frac{\overline{x_0}}{|x_0|}x_0) - x_1\frac{\overline{x_0}}{|x_0|}\epsilon_0}{|x_0|^2+\epsilon_0}
\end{aligned}
$$

and use the fact that $y_0 = \sqrt{|x_0|^2 + \epsilon_0}$ and $y_0 - \frac{\overline{x_0}}{|x_0|}x_0 = \frac{\epsilon_0}{2\sqrt{\xi}}$ to get

$$
\begin{aligned}
=& \frac{\frac{(1-i)\epsilon_0}{2}\sqrt{|x_0|^2+\epsilon_0} + \overline{x_0}x_1\frac{\epsilon_0}{2\sqrt{\xi}} - x_1\frac{\overline{x_0}}{|x_0|}\epsilon_0}{|x_0|^2+\epsilon_0} \\
=& \left(\frac{\frac{1}{2}(1-i)\sqrt{|x_0|^2+\epsilon_0} + \overline{x_0}x_1\left(\frac{1}{2\sqrt{\xi}} - \frac{1}{|x_0|}\right)}{|x_0|^2+\epsilon_0}\right)\epsilon_0\,,
\end{aligned}
$$

which proves the base case.

For the $k$th (with $2 < k < d$) inductive step, we assume that we have obtained $y_k$ with the given information such that $y_k - \frac{\overline{x_0}}{|x_0|}x_k = \frac{\overline{x_1}x_k}{|x_1|^2}\left(\frac{\frac{1}{2}(1-i)\sqrt{|x_0|^2+\epsilon_0}+\overline{x_0}x_1\left(\frac{1}{2\sqrt{\xi}}-\frac{1}{|x_0|}\right)}{|x_0|^2+\epsilon_0}\right)\epsilon_0.$

Then because $(t_k - \overline{x_k}x_{k+1}) = 0$ and $\epsilon_k = 0$, we get

$$
\begin{aligned}
y_{k+1} - \frac{\overline{x_0}}{|x_0|}x_{k+1} =& \frac{(t_k - \overline{x_k}x_{k+1})y_k + \overline{x_k}x_{k+1}(y_k - \frac{\overline{x_0}}{|x_0|}x_k) - x_{k+1}\frac{\overline{x_0}}{|x_0|}\epsilon_k}{|x_k|^2 + \epsilon_k} \\
=& \frac{\overline{x_k}x_{k+1}(y_k - \frac{\overline{x_0}}{|x_0|}x_k)}{|x_k|^2} \\
=& \frac{\overline{x_k}x_{k+1}}{|x_k|^2}\frac{\overline{x_1}x_k}{|x_1|^2}\left(\frac{\frac{1}{2}(1-i)\sqrt{|x_0|^2+\epsilon_0} + \overline{x_0}x_1\left(\frac{1}{2\sqrt{\xi}} - \frac{1}{|x_0|}\right)}{|x_0|^2+\epsilon_0}\right)\epsilon_0 \\
=& \frac{\overline{x_1}x_{k+1}}{|x_1|^2}\left(\frac{\frac{1}{2}(1-i)\sqrt{|x_0|^2+\epsilon_0} + \overline{x_0}x_1\left(\frac{1}{2\sqrt{\xi}} - \frac{1}{|x_0|}\right)}{|x_0|^2+\epsilon_0}\right)\epsilon_0
\end{aligned}
$$

which proves the claim. □

Now we establish a lower bound on the error of each component of this pathological vector.

**Corollary 2.6.10.** *Let* $\alpha_0 \in \left(\frac{1}{4}, 1\right)$ *and* $\alpha_1 \in (0,1)$. *For any* $x \in \mathbb{C}^d$, *if* $\epsilon \in \mathbb{R}^{3d-2}$ *such that* $\epsilon_0 < 0$ *and all other components* $\epsilon_j = 0$ *and if* $\{e_j\}_{j=0}^{d-1}$ *is an orthonormal basis such that* $x$ *is full with respect to* $\{e_j\}_{j=0}^{d-1}$, *and* $|\langle x, e_0\rangle| \leq \sqrt{2}\left(1 - \frac{1}{2\sqrt{\alpha_0}}\right)\alpha_1|\langle x, e_1\rangle|$, *and* $\epsilon_0 \geq -(1-\alpha_0)|\langle x, e_0\rangle|^2$, *then a vector* $y$ *may be obtained such that*

$$\left|y_0 - \frac{\overline{x_0}}{|x_0|}x_0\right| \geq \frac{|\epsilon_0|}{2|x_0|}$$

*and for all* $k$ *from* $1$ *to* $d-1$,

$$\left|y_k - \frac{\overline{x_0}}{|x_0|}x_k\right| \geq \frac{|x_k|}{|x_0|}\left(1 - \frac{1}{2\sqrt{\alpha_0}}\right)(1-\alpha_1)\frac{|\epsilon_0|}{|x_0|}$$

*by using the generalized step 3 algorithm presented in section 2.4 with the phaseless measurements*

$$\{|\langle x, e_k\rangle|^2 + \epsilon_k\}_{k=0}^{d-1}$$

$$\{|\langle x, e_k - e_{k+1}\rangle|^2 + \epsilon_{k+d}\}_{k=0}^{d-2}$$

$$\{|\langle x, e_k - ie_{k+1}\rangle|^2 + \epsilon_{k+2d-1}\}_{k=0}^{d-2}\,.$$

*Proof.* By the previous Lemma, we know that there exists a $\xi$ such that $|x_0|^2 + \epsilon_0 \le \xi \le |x_0|^2$, and

$$y_0 - \frac{\overline{x_0}}{|x_0|}x_0 = \frac{\epsilon_0}{2\sqrt{\xi}}$$

and for all $k$ from 1 to $d-1$,

$$y_k - \frac{\overline{x_0}}{|x_0|}x_k = \frac{\overline{x_1}x_k}{|x_1|^2}\left(\frac{\frac{1}{2}(1-i)\sqrt{|x_0|^2+\epsilon_0} + \overline{x_0}x_1\left(\frac{1}{2\sqrt{\xi}} - \frac{1}{|x_0|}\right)}{|x_0|^2+\epsilon_0}\right)\epsilon_0\,.$$

Thus,

$$\left|y_0 - \frac{\overline{x_0}}{|x_0|}x_0\right| = \frac{|\epsilon_0|}{2\sqrt{\xi}} \ge \frac{|\epsilon_0|}{2|x_0|}$$

and for all $k$ from 1 to $d-1$,

$$\begin{aligned}\left|y_k - \frac{\overline{x_0}}{|x_0|}x_k\right| &= \frac{|x_1||x_k|}{|x_1|^2}\left|\frac{\frac{1}{2}(1-i)\sqrt{|x_0|^2+\epsilon_0} + \overline{x_0}x_1\left(\frac{1}{2\sqrt{\xi}} - \frac{1}{|x_0|}\right)}{|x_0|^2+\epsilon_0}\right||\epsilon_0| \\ &= \frac{|x_k|}{|x_1|}\left|\frac{\frac{1}{2}(1-i)\sqrt{|x_0|^2+\epsilon_0} + \overline{x_0}x_1\left(\frac{1}{2\sqrt{\xi}} - \frac{1}{|x_0|}\right)}{|x_0|^2+\epsilon_0}\right||\epsilon_0|\,.\end{aligned}$$

Note that

$$\frac{1}{|x_0|} - \frac{1}{2\sqrt{\xi}} \geq \frac{1}{|x_0|} - \frac{1}{2\sqrt{|x_0|^2 + \epsilon_0}} \geq \frac{1}{|x_0|} - \frac{1}{2\sqrt{\alpha_0 |x_0|^2}} = \frac{1}{|x_0|}\left(1 - \frac{1}{2\sqrt{\alpha_0}}\right) > 0\,.$$

By the above equation and the assumption that $\epsilon_0 < 0$,

$$\begin{aligned}
\left|\overline{x_0} x_1 \left(\frac{1}{|x_0|} - \frac{1}{2\sqrt{\xi}}\right)\right| - \left|\tfrac{1}{2}(1-i)\sqrt{|x_0|^2 + \epsilon_0}\right| &\geq |x_0||x_1| \frac{1}{|x_0|}\left(1 - \frac{1}{2\sqrt{\alpha_0}}\right) - \tfrac{\sqrt{2}}{2}|x_0| \\
&= |x_1|\left(1 - \frac{1}{2\sqrt{\alpha_0}} - \frac{\sqrt{2}}{2}\frac{|x_0|}{|x_1|}\right)
\end{aligned}$$

and using the assumption on the ratio $\frac{|x_0|}{|x_1|}$ gives

$$\begin{aligned}
&\geq |x_1|\left(1 - \frac{1}{2\sqrt{\alpha_0}} - \left(1 - \frac{1}{2\sqrt{\alpha_0}}\right)\alpha_1\right) \\
&= \left(1 - \frac{1}{2\sqrt{\alpha_0}}\right)(1-\alpha_1)|x_1| \\
&> 0\,.
\end{aligned}$$

Thus, by the reverse triangle inequality

$$\begin{aligned}
\left| y_k - \frac{\overline{x_0}}{|x_0|} x_k \right| &= \frac{|x_k|}{|x_1|} \frac{\left|\frac{1}{2}(1-i)\sqrt{|x_0|^2 + \epsilon_0} + \overline{x_0} x_1 \left(\frac{1}{2\sqrt{\xi}} - \frac{1}{|x_0|}\right)\right|}{|x_0|^2 + \epsilon_0} |\epsilon_0| \\
&\geq \frac{|x_k|}{|x_1|} \frac{\left|\left|\overline{x_0} x_1 \left(\frac{1}{|x_0|} - \frac{1}{2\sqrt{\xi}}\right)\right| - \left|\frac{1}{2}(1-i)\sqrt{|x_0|^2 + \epsilon_0}\right|\right|}{|x_0|^2 + \epsilon_0} |\epsilon_0| \\
&\geq \frac{|x_k|}{|x_1|} \frac{\left(1 - \frac{1}{2\sqrt{\alpha_0}}\right)(1-\alpha_1)|x_1|}{|x_0|^2 + \epsilon_0} |\epsilon_0| \\
&\geq \frac{|x_k|}{|x_0|}\left(1 - \frac{1}{2\sqrt{\alpha_0}}\right)(1-\alpha_1)\frac{|\epsilon_0|}{|x_0|}
\end{aligned}$$

□

We remove the dependence on the unknown $\alpha_0$ and $\alpha_1$ on lower bound on the error of each component of this pathological vector, by assuming a ration between the first two components of $x$.

**Corollary 2.6.11.** *For any $x \in \mathbb{C}^d$, if $\epsilon \in \mathbb{R}^{3d-2}$ such that $\epsilon_0 < 0$ and all other components $\epsilon_j = 0$ and if $\{e_j\}_{j=0}^{d-1}$ is an orthonormal basis such that $x$ is full with respect to $\{e_j\}_{j=0}^{d-1}$, and $|\langle x, e_0\rangle| \leq \frac{1}{2\sqrt{2}}\left(\sqrt{2}-1\right)^2 |\langle x, e_1\rangle|$, and $\epsilon_0 \geq -\frac{1}{2}|\langle x, e_0\rangle|^2$, then a vector $y$ may be obtained such that*

$$\left|y_0 - \frac{\overline{x_0}}{|x_0|}x_0\right| \geq \frac{|\epsilon_0|}{2|x_0|}$$

*and for all $k$ from $1$ to $d-1$,*

$$\left|y_k - \frac{\overline{x_0}}{|x_0|}x_k\right| \geq \frac{1}{4}\frac{|x_k|}{|x_0|}\frac{|\epsilon_0|}{|x_0|}$$

*by using the generalized step 3 algorithm presented in section 2.4 with the phaseless measurements*

$$\{|\langle x, e_k\rangle|^2 + \epsilon_k\}_{k=0}^{d-1}$$
$$\{|\langle x, e_k - e_{k+1}\rangle|^2 + \epsilon_{k+d}\}_{k=0}^{d-2}$$
$$\{|\langle x, e_k - ie_{k+1}\rangle|^2 + \epsilon_{k+2d-1}\}_{k=0}^{d-2}.$$

*Proof.* Plug in $\alpha_0 = \frac{1}{2}$ and $\alpha_1 = \frac{\sqrt{2}-1}{2\sqrt{2}}$ into the previous corollary. $\square$

We sum the squares of the values in the above lemma to obtain a lower bound on the $\ell_2$ norm of the difference between the recovered vector and the original signal.

**Corollary 2.6.12.** *For any $x \in \mathbb{C}^d$, if $\epsilon \in \mathbb{R}^{3d-2}$ such that $\epsilon_0 < 0$ and all other components $\epsilon_j = 0$ and if $\{e_j\}_{j=0}^{d-1}$ is an orthonormal basis such that $x$ is full with respect to $\{e_j\}_{j=0}^{d-1}$, and*

$|\langle x, e_0\rangle| \leq \frac{1}{2\sqrt{2}}\left(\sqrt{2}-1\right)^2 |\langle x, e_1\rangle|$, *and* $\epsilon_0 \geq -\frac{1}{2}|\langle x, e_0\rangle|^2$, *then a vector* $y$ *may be obtained such that*

$$\left\| y - \frac{\overline{x_0}}{|x_0|}x \right\|_2 \geq \frac{1}{4}\frac{\|x\|_2}{|x_0|}\frac{|\epsilon_0|}{|x_0|}$$

*by using the generalized step 3 algorithm presented in section 2.4 with the phaseless measurements*

$$\begin{aligned}\{|\langle x, e_k\rangle|^2 + \epsilon_k\}_{k=0}^{d-1}\\ \{|\langle x, e_k - e_{k+1}\rangle|^2 + \epsilon_{k+d}\}_{k=0}^{d-2}\\ \{|\langle x, e_k - ie_{k+1}\rangle|^2 + \epsilon_{k+2d-1}\}_{k=0}^{d-2}.\end{aligned}$$

*If, in addition, we have* $|\langle x, e_1\rangle| \leq \frac{1}{C}\|x\|_2$, *then*

$$\left\| y - \frac{\overline{x_0}}{|x_0|}x \right\|_2 \geq \frac{C^2}{4}\frac{|\epsilon_0|}{\|x\|_2}$$

*Proof.* By the previous corollary

$$\left\| y - \frac{\overline{x_0}}{|x_0|}x \right\|_2 = \sqrt{\sum_{k=0}^{d-1}\left| y_k - \tfrac{\overline{x_0}}{|x_0|}x_k\right|^2} \geq \sqrt{\sum_{k=0}^{d-1}\left(\frac{1}{4}\frac{|x_k|}{|x_0|}\frac{|\epsilon_0|}{|x_0|}\right)^2} = \frac{1}{4}\frac{\|x\|_2}{|x_0|}\frac{|\epsilon_0|}{|x_0|}$$

□

**Theorem 2.6.13.** *Let* $d$ *be an even integer with* $d > 11$ *and let* $p \in \mathcal{P}_d$ *be a polynomial with roots*

$$z_k = e^{\frac{i\pi}{d}\cos\left(\frac{(2k-1)\pi}{2(d-1)}\right)}.$$

*Then* $p(z) = a\prod_{k=1}^{d-1}(z - z_k)$ *for some* $a \in \mathbb{C}$. *Let* $\alpha \in (0, \frac{1}{2}|p(1)|^2]$, $N \geq d$, *and* $\omega_{2N-1}$ *be the* $2N-1$ *root of unity. For* $z_0 = 1$ *and noise vectors* $\eta_0, \eta_1, \eta_2 \in \mathbb{R}^{2N-1}$ *such that*

$\eta_0 = \left(-\alpha|K_1(\omega_{2N-1}^k)|^2\right)_{k=0}^{2N-2}$ *and* $\eta_1 = \eta_2 = 0$*, if* $C = \frac{\sqrt{d}\|p\|_2}{|p(1)|}$*, then an approximation* $\tilde{p} \in \mathcal{P}_d$ *can be obtained by using the algorithm given in section 2.3 with the phaseless measurements*

$$\{|p|^2(\omega_{2N-1}^k) + (\eta_0)_k\}_{k=0}^{2N-2}$$

$$\{|(I - R_{0,\omega_d})p|^2(\omega_{2N-1}^k) + (\eta_1)_k\}_{k=0}^{2N-2}$$

$$\{|(I - iR_{0,\omega_d})p|^2(\omega_{2N-1}^k) + (\eta_2)_k\}_{k=0}^{2N-2}$$

*such that for some* $c_0 \in \mathbb{T}$

$$\|\tilde{p} - c_0 p\|_2 \geq \frac{C^2\sqrt{3}}{4d\sqrt{(2N-1)(2d^3+d)}} \frac{\|\eta_0\|_2}{\|p\|_2}\,.$$

*Proof.* Note that any trigonometric polynomial in $\mathcal{T}_d$ may be recovered using Dirichlet kernel interpolation on $\mathcal{T}_N$ for any $N \geq d$, as shown in Lemma 2.5.2. This requires the use of point evaluations at $\{\omega_{2N-1}^j\}_{j=0}^{2N-2}$. Note that $|p|^2$, $|(I - R_{0,\omega_d})p|^2$, and $|(I - iR_{0,\omega_d})p|^2$ are in $\mathcal{T}_d$, and using the above, these functions may be interpolated from the perturbed values at $\{\omega_{2N-1}^j\}_{j=0}^{2N-2}$. The approximating trigonometric polynomials obtained from this interpolation are $f_0 = \sum_{j=0}^{2N-2}(|p(\omega_{2N-1}^j)|^2 + \eta_0)\frac{1}{2N-1}D_{\omega_{2N-1}^j,d-1}$, $f_1 = |(I - R_{0,\omega_d})p|^2$, and $f_2 = |(I - iR_{0,\omega_d})p|^2$. then

$$f_0 - |p|^2 = \sum_{j=0}^{2N-2}(\eta_0)_j \frac{1}{2N-1} D_{\omega_{2N-1}^j,d-1} = -\alpha|K_1|^2\,.$$

The set $\{\frac{1}{\sqrt{d}}K_{z_0\omega_d^j}\}_{j=0}^{d-1}$ is an ordered orthonormal basis for $\mathcal{P}_d$. Let $x$ be the vector $p$ represented in this ordered basis. Then for any $j$ from 0 to $d-1$,

$$|x_j|^2 = \left|\left\langle p, \frac{1}{\sqrt{d}}K_{z_0\omega_d^j}\right\rangle\right|^2 = \frac{1}{d}|\langle p, K_{z_0\omega_d^j}\rangle|^2 = \frac{1}{d}|p(z_0\omega_d^j)|^2\,.$$

Then by Corollary 2.6.8, we know that

$$\frac{|x_0|}{|x_1|} = \frac{|p(1)|}{|p(e^{\frac{2i\pi}{d}})|} \leq \frac{(\sqrt{2}-1)^2}{2\sqrt{2}}$$

If we define $\epsilon \in \mathbb{R}^{3d-2}$ such that

$$\epsilon_j = \begin{cases} \frac{1}{d}\left(|f_0(z_0\omega_d^j)| - |p(z_0\omega_d^j)|^2\right) & \text{if } 0 \leq j \leq d-1 \\ \frac{1}{d}\left(|f_1(z_0\omega_d^j)| - |p(z_0\omega_d^j) - p(z_0\omega_d^{j+1})|^2\right) & \text{if } d \leq j \leq 2d-2 \\ \frac{1}{d}\left(|f_2(z_0\omega_d^{j+1})| - |p(z_0\omega_d^{j+1}) - ip(z_0\omega_d^{j+2})|^2\right) & \text{if } 2d-1 \leq j \leq 3d-3 \end{cases}$$

then $\epsilon_0 = -\frac{\alpha}{d} \geq -\frac{1}{2}|x_0|^2$ and all other $\epsilon_j = 0$. Note that

$$\|\eta_0\|_2 = \alpha\sqrt{\sum_{k=1}^{2N-1} |K_1(\omega_{2N-1}^k)|^4} = \alpha\sqrt{2N-1}\,\left\||K_1|^2\right\|_2 = \frac{\alpha}{\sqrt{3}}\sqrt{(2N-1)(2d^3+d)}\,.$$

Then by the corollary 2.6.12, the algorithm from section 2.4 produces a vector $y$ such that

$$\left\|y - \frac{\overline{x_0}}{|x_0|}x\right\|_2 \geq \frac{C^2}{4}\frac{|\epsilon_0|}{\|x\|_2} = \frac{C^2\sqrt{3}}{4d\sqrt{(2N-1)(2d^3+d)}}\frac{\|\eta_0\|_2}{\|p\|_2}$$

Then, if we define $\tilde{p} = \sum_{k=0}^{d-1} y_k \frac{1}{\sqrt{d}} K_{\tilde{z}_0\omega_d^k}$,

$$\begin{aligned}
\left\|\tilde{p} - \frac{\overline{p(1)}}{|p(1)|}p\right\|_2 &= \left\|\sum_{k=0}^{d-1} y_k \frac{1}{\sqrt{d}} K_{\tilde{z}_0\omega_d^k} - \frac{\overline{p(1)}}{|p(1)|}\sum_{k=0}^{d-1} p(\tilde{z}_0\omega_d^k)\frac{1}{\sqrt{d}}K_{\tilde{z}_0\omega_d^k}\right\|_2 \\
&= \left\|\sum_{k=0}^{d-1}\left(y_k - \frac{\overline{p(1)}}{|p(1)|}p(\tilde{z}_0\omega_d^k)\right)\frac{1}{\sqrt{d}}K_{\tilde{z}_0\omega_d^k}\right\|_2 \\
&= \left\|y - \frac{\overline{x_0}}{|x_0|}x\right\|_2 \\
&\geq \frac{C^2\sqrt{3}}{4d\sqrt{(2N-1)(2d^3+d)}}\frac{\|\eta_0\|_2}{\|p\|_2}\,. \qquad \square
\end{aligned}$$

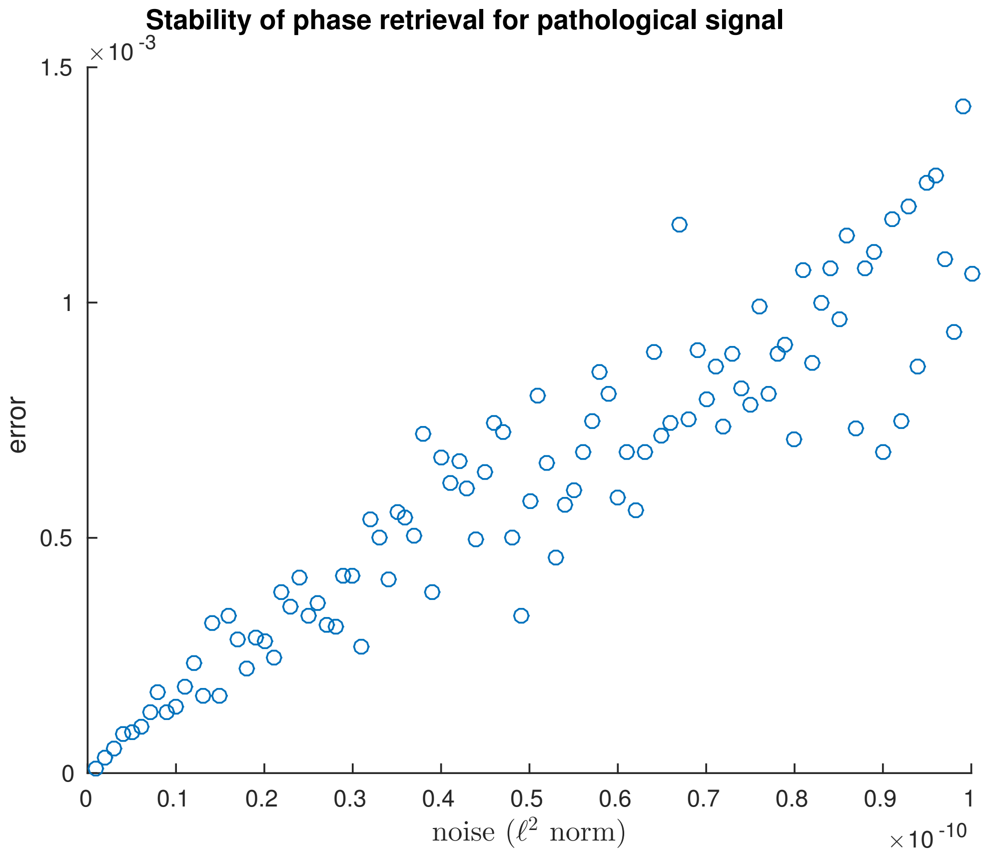


Figure 2.2: The $L^2$ distance between the signal recovered by the phase retrieval algorithm and the actual signal for the worst-case polynomial in $d = 7$, with no oversampling, as a function of input noise. Each point represents the worst error out of 100 random noise vectors of the chosen norm.

This result gives us the desired lower bound on the recovery error for our pathological signal. Figure 2.2 shows the error resulting from attempting to reconstruct the pathological signal using the algorithm. The behavior is still linear, as predicted, but the algorithm magnifies the noise by a factor of $10^7$. Note that the behavior observed in the case of a pathological signal is very abnormal behavior. In fact, in experiments, randomly chosen signals reduced noise of the magnitude depicted here to the square root of machine precision.

## 2.7 Mean squared error

Much of the above work dealt with calculating what happens when the input error is the worst-possible error of any particular norm, or close to it, and when we are using the polynomial that has the worst interaction with such an error. For a more typical case, we calculate the mean squared error for a fixed polynomial. We still use the worst-case polynomial for this. To calculate the mean squared error, we need a precise value for the $\ell_1$ norm of the weight vector.

**Lemma 2.7.1.** *Let $\tilde{m} \in (0,1]$. Define $C(t_1,t_2) \in \mathbb{R}_+$ by $C(t_1,t_2) = \frac{(1+\sqrt{2})\sqrt{2d-1}t_1 + d\|p\|_2^2}{t_2\sqrt{d}\|p\|_2^2}$ and $w(C,u) \in \mathbb{R}^{3d-2}$ by*

$$w(C,u)_j = \begin{cases} \sum_{k=1}^{d-1} \frac{C^{k-1}(1+\frac{\sqrt{2}}{2})}{u} + \sum_{k=0}^{d-1} \frac{C^k}{2\sqrt{u}} & \text{if } j = 0 \\ \sum_{k=1}^{d-j-1} \frac{C^{k-1}(1+\frac{\sqrt{2}}{2})}{u} + \sum_{k=0}^{d-j-1} \frac{C^k\frac{\sqrt{2}}{2}}{u} & \text{if } 1 \le j \le d-1 \\ \sum_{k=1}^{2d-1-j} \frac{C^{k-1}}{2u} & \text{if } d \le j \le 2d-2 \\ \sum_{k=1}^{3d-2-j} \frac{C^{k-1}}{2u} & \text{if } 2d-1 \le j \le 3d-3 \end{cases}.$$

*If for each $k$ from $0$ to $2d-2$, $J(k) = d - |d-1-k|$, $B_1(k) = \left(\frac{1}{2}\sqrt{\tilde{m}} + (d-k)(2+\sqrt{2})\right)$,* *and* $B_2(k) = \sum_{j=1}^{J(k)} B_1(k-j+2)B_1(j)$, *then*

$$\|w(C(t_1,\tilde{m}),\tilde{m})\|_1^2 = \sum_{n=0}^{2d-2}\left(\sum_{k=n}^{2d-2}\frac{1}{\tilde{m}^{k+2}}\binom{k}{n}\left((1+\sqrt{2})\sqrt{2-\tfrac{1}{d}}\right)^n d^{\frac{k-n}{2}}B_2(k)\right)\left(\frac{t_1}{\|p\|_2^2}\right)^n.$$

*Proof.* Note that by rearranging terms in the $\ell_1$ norm

$$\begin{aligned}
\|w(C,u)\|_1 =& \sum_{k=1}^{d-1}\frac{C^{k-1}(1+\frac{\sqrt{2}}{2})}{u} + \sum_{k=0}^{d-1}\frac{C^k}{2\sqrt{u}} \\
&+ \sum_{j=1}^{d-1}\left(\sum_{k=1}^{d-j-1}\frac{C^{k-1}(1+\frac{\sqrt{2}}{2})}{u} + \sum_{k=0}^{d-j-1}\frac{C^k\frac{\sqrt{2}}{2}}{u}\right) \\
&+ 2\sum_{j=0}^{d-2}\sum_{k=1}^{d-j-1}\frac{C^{k-1}}{2u} \\
=& \sum_{k=0}^{d-1}\frac{C^k}{2\sqrt{u}} + \sum_{j=0}^{d-1}\sum_{k=1}^{d-j-1}\frac{C^{k-1}(2+\frac{\sqrt{2}}{2})}{u} \\
&+ \sum_{j=1}^{d-1}\sum_{k=0}^{d-j-1}\frac{C^k\frac{\sqrt{2}}{2}}{u} \\
=& \sum_{k=0}^{d-1}\frac{C^k}{2\sqrt{u}} + \sum_{k=1}^{d-1}\sum_{j=1}^{d-k}\frac{C^{k-1}(2+\sqrt{2})}{u} \\
=& \sum_{k=1}^{d}C^{k-1}\left(\frac{1}{2\sqrt{u}} + \frac{(d-k)(2+\sqrt{2})}{u}\right) \\
=& \frac{1}{u}\sum_{k=1}^{d}C^{k-1}B_1(k)
\end{aligned}$$

Then, if we square both sides

$$
\begin{aligned}
\|w(C,u)\|_1^2 =& \frac{1}{u^2}\sum_{k=1}^{d}\sum_{j=0}^{d-1} C^{j+k-2}B_1(k)B_1(j) \\
=& \frac{1}{u^2}\sum_{k=0}^{2d-2}\sum_{j=1}^{J(k)} C^{k}B_1(k-j+2)B_1(j) \\
=& \frac{1}{u^2}\sum_{k=0}^{2d-2} C^{k}B_2(k)
\end{aligned}
$$

and after plugging in the formula for $C(t_1, \tilde{m})$ in place of $C$, and $\tilde{m}$ in place of $u$

$$
= \frac{1}{\tilde{m}^2}\sum_{k=0}^{2d-2}\left(\frac{(1+\sqrt{2})\sqrt{2d-1}t_1 + d\|p\|_2^2}{\tilde{m}\sqrt{d}\|p\|_2^2}\right)^k B_2(k)
$$

and using the binomial theorem

$$
\begin{aligned}
=& \sum_{k=0}^{2d-2}\frac{1}{\tilde{m}^{k+2}}\sum_{n=0}^{k}\binom{k}{n}\left(\frac{(1+\sqrt{2})\sqrt{2-\frac{1}{d}}t_1}{\|p\|_2^2}\right)^n d^{\frac{k-n}{2}}B_2(k) \\
=& \sum_{n=0}^{2d-2}\sum_{k=n}^{2d-2}\frac{1}{\tilde{m}^{k+2}}\binom{k}{n}\left(\frac{(1+\sqrt{2})\sqrt{2-\frac{1}{d}}t_1}{\|p\|_2^2}\right)^n d^{\frac{k-n}{2}}B_2(k)\,.
\end{aligned}
$$

□

With the above bound on the norm of the weight vector, we can begin to calculate the mean squared error. Here, we calculate the mean squared error conditioned on a lower bound for the values of the perturbed polynomial.

**Theorem 2.7.2.** *Let $\tilde{m} \in (0,1]$ and $\sigma > 0$. For any nonzero $p \in \mathcal{P}_d$, and any noise vectors $\eta_0, \eta_1, \eta_2 \in \mathbb{R}^{2N-1}$ composed of i.i.d. Gaussian entries with variance $\sigma^2$, if there exists a $z_0 \in \mathbb{T}$ such that $\min\{|p(z_0\omega_d^j)|^2 + \sum_{k=1}^{2N-1}\eta_{0,k}\frac{1}{2N-1}D_{\omega_{2N-1}^k, d-1}(z_0\omega_d^j)\}_{j=0}^{d-1} \geq \tilde{m}d\|p\|_2^2$ and*

*$|p(z_0\omega_d)|^2 \geq d\tilde{m}\|p\|_2^2$, then an approximation $\tilde{p} \in \mathcal{P}_d$ can be constructed using the Dirichlet Kernel and the the phaseless measurements*

$$\{|p|^2(\omega_{2N-1}^k) + (\eta_0)_k\}_{k=0}^{2N-2}$$
$$\{|(I - R_{0,\omega_d})p|^2(\omega_{2N-1}^k) + (\eta_0)_k\}_{k=0}^{2N-2}$$
$$\{|(I - iR_{0,\omega_d})p|^2(\omega_{2N-1}^k) + (\eta_0)_k\}_{k=0}^{2N-2}$$

*such that if $C(t_1, t_2)$, $J(k)$, $B_1(k)$, and $B_2(k)$ are as defined in the previous lemma, then for some $c_0 \in \mathbb{T}$*

$$\mathbb{E}[\|\tilde{p} - c_0 p\|_2^2] \leq 2\left(\sum_{k=0}^{2d-2} \frac{1}{\tilde{m}^{k+2}} d^{\frac{k}{2}} B_2(k)\right) \frac{6d-3}{2N-1} \frac{\sigma^2}{\|p\|_2^2} + O\left(\left(\frac{\sigma}{\sqrt{2N-1}}\right)^3\right).$$

*Proof.* Let $m \in (0,1]^d$ such that each entry $m_j = \tilde{m}$. Note any trigonometric polynomial in $\mathcal{T}_d$ may be recovered using Dirichlet kernel interpolation on $\mathcal{T}_N$ for any $N \geq d$, as shown in Lemma 2.5.2. This requires the use of point evaluations at $\{\omega_{2N-1}^j\}_{j=0}^{2N-2}$. Note that $|p|^2$, $|(I - R_{0,\omega_d})p|^2$, and $|(I - iR_{0,\omega_d})p|^2$ are in $\mathcal{T}_d$, and using the above , these functions may be interpolated from the values at $\{\omega_{2N-1}^j\}_{j=0}^{2N-2}$. Approximating trigonometric polynomials $f_0 \approx |p|^2$, $f_1 \approx |(I - R_{0,\omega_d})p|^2$, and $f_2 \approx |(I - iR_{0,\omega_d})p|^2$ are obtained from this interpolation. Then by Lemma 2.5.3

$$\mathbb{E}[\|f_0 - |p|^2\|_2^2] = \frac{2d-1}{(2N-1)^2}\mathbb{E}[\|\eta_0\|_2^2]$$

$$\mathbb{E}[\|f_1 - |(I - R_{0,\omega_d})p|^2\|_2^2] = \frac{2d-1}{(2N-1)^2}\mathbb{E}[\|\eta_1\|_2^2]$$

and

$$\mathbb{E}[\|f_2 - |(I - iR_{0,\omega_d})p|^2\|_2^2] = \frac{2d-1}{(2N-1)^2}\mathbb{E}[\|\eta_2\|_2^2].$$

We may apply lemma 2.6.1 to get an approximation $\tilde{p} \in \mathcal{P}_d$ such that for some $c_0 \in \mathbb{T}$, if

$E = \{\|f_0 - |p|^2\|_2, \|f_1 - |(I - R_{0,\omega_d})p|^2\|_2, \|f_2 - |(I - iR_{0,\omega_d})p|^2\|_2\}$, then

$$\|\tilde{p} - c_0 p\|_2 \leq \|w(C(\|E\|_\infty, \min(m)), \min(m))\|_2 \frac{\sqrt{2}\|E\|_2}{\|p\|_2}$$

and because all components of $w$ are increasing in $C$,

$$\leq \|w(C(\|E\|_2, \tilde{m}), \tilde{m})\|_2 \frac{\sqrt{2}\|E\|_2}{\|p\|_2}$$

and by equivalence of norms

$$\leq \|w(C(\|E\|_2, \tilde{m}), \tilde{m})\|_1 \frac{\sqrt{2}\|E\|_2}{\|p\|_2}.$$

Then, by squaring both sides, and taking the expectation, we get

$$\mathbb{E}[\|\tilde{p} - c_0 p\|_2^2] \leq \mathbb{E}\left[\|w(C(\|E\|_2, \tilde{m}), m)\|_1^2 \frac{2\|E\|_2^2}{\|p\|_2^2}\right]$$

and by Lemma 2.7.1

$$= 2\sum_{n=0}^{2d-2}\left(\sum_{k=n}^{2d-2} \frac{1}{\tilde{m}^{k+2}}\binom{k}{n}\left((1+\sqrt{2})\sqrt{2-\tfrac{1}{d}}\right)^n d^{\frac{k-n}{2}} B_2(k)\right)\frac{\mathbb{E}[\|E\|_2^{n+2}]}{\|p\|_2^{2n+2}}$$

Note that $\mathbb{E}[\|E\|_2^2] = \frac{2d-1}{(2N-1)^2}\mathbb{E}[\|\eta_0\|_2^2 + \|\eta_1\|_2^2 + \|\eta_2\|_2^2] = 3\frac{2d-1}{2N-1}\sigma^2$ and that $\|E\|_2$ has a scaled Chi distribution with $6d-3$ degrees of freedom, so that

$$\mathbb{E}[\|E\|_2^{n+2}] = 2^{\frac{n}{2}+1}\frac{\Gamma(\frac{6d-3+n+2}{2})}{\Gamma(\frac{6d-3}{2})}\left(\frac{\sigma}{\sqrt{2N-1}}\right)^{n+2} \leq (6d-3+n)^{\frac{n+2}{2}}\left(\frac{\sigma}{\sqrt{2N-1}}\right)^{n+2}$$

Thus,

$$\mathbb{E}[\|\tilde{p} - c_0 p\|_2^2] \leq 2 \sum_{n=0}^{2d-2} \left( \sum_{k=n}^{2d-2} \frac{1}{\tilde{m}^{k+2}} \binom{k}{n} \left( (1+\sqrt{2})\sqrt{2 - \frac{1}{d}} \right)^n d^{\frac{k-n}{2}} B_2(k) \right) \frac{\mathbb{E}[\|E\|_2^{n+2}]}{\|p\|_2^{2n+2}}$$
$$\leq 2 \left( \sum_{k=0}^{2d-2} \frac{1}{\tilde{m}^{k+2}} d^{\frac{k}{2}} B_2(k) \right) \frac{(6d-3)\left(\frac{\sigma}{\sqrt{2N-1}}\right)^2}{\|p\|_2^2} + O\left( \left( \frac{\sigma}{\sqrt{2N-1}} \right)^3 \right).$$

□

This conditional mean squared error is not useful without knowing the probability of the conditional statement being true. Thus, we establish an upper bound on the falsehood of the statement.

**Theorem 2.7.3.** *Let* $r = \sin(\frac{2\pi}{(d-1)d^2})$, $\beta = \frac{r^{\frac{(d-1)d}{2}} \left(\frac{d-1}{2d}\right)^d \frac{2}{d-1}}{\left(\prod_{k=1}^{d-1}(r^k+1)\right)}$, $L > 0$, *and* $\sigma > 0$ *such that* $\sigma < \frac{\sqrt[4]{e}\beta^2 L^2}{\sqrt{2d-1}}$. *Let* $\tilde{m} = \frac{1}{d}\left( \beta^2 - \frac{1}{L^2}\sqrt{\frac{(2d-1)\sigma^2}{2N-1} W((2N-1)^2)} \right)$, *where* $W$ *is the Lambert W function, which is the inverse of the function* $g(x) = xe^x$. *Note that* $0 < \tilde{m} < \frac{\beta^2}{d}$. *For any* $p \in \mathcal{P}_d$ *with* $\|p\|_2 \geq L$, *and any noise vectors* $\eta_0, \eta_1, \eta_2 \in \mathbb{R}^{2N-1}$ *composed of i.i.d. Gaussian entries with variance* $\sigma^2$, *if we define* $f \in \mathcal{T}_d$ *by* $f = \sum_{k=1}^{2N-1} \eta_{0,k} \frac{1}{2N-1} D_{\omega_{2N-1}^k, d-1}$ *and let* $F$ *denote the event that there does not exist a* $z$ *on the unit circle such that* $|p(z\omega_d)|^2 \geq \tilde{m}d\|p\|_2^2$ *and* $\min\{|p(z\omega_d^j)|^2 + f(z\omega_d^j)\}_{j=0}^{d-1} \geq \tilde{m}d\|p\|_2^2$, *then*

$$Pr(F) \leq \frac{d}{(2N-1)\sqrt{2\pi}}$$

*Proof.* Note that the probability of every $z$ on the unit circle satisfying the equation

$$\min \left\{ |p(z\omega_d^j)|^2 + f(z\omega_d^j) \right\}_{j=0}^{d-1} < \tilde{m}d\|p\|_2^2$$

is less than the probability of this being true for the value of $z_0$ given by Lemma A.1.3.

Then, if we let $F_j$ be the event that

$$|p(z_0\omega_d^j)|^2 + f(z_0\omega_d^j) < \tilde{m}d\|p\|_2^2$$

we get that

$$Pr(F) \le Pr(\cup_{j=0}^{d-1} F_j) \le \sum_{j=0}^{d-1} Pr(F_j)\,.$$

By Lemma A.1.4 we know that for all $j$ from 1 to $d$, $|p(z_0\omega_d^j)| \ge \beta\|p\|_1 \ge \beta\|p\|_2$ where we define $\|p\|_1$ to be the $\ell_1$ norm of the monomial coefficients of $p$. Also note that $f$ has i.i.d. Gaussian entries with variance $\frac{\sigma^2}{2N-1}$, so for each $j$ from 0 to $d-1$, $f(z_0\omega_d^j)$ is Gaussian with variance $\frac{(2d-1)\sigma^2}{2N-1}$. Thus,

$$\begin{aligned}
Pr(F_j) =& Pr(|p(z_0\omega_d^j)|^2 + f(z_0\omega_d^j) < \tilde{m}d\|p\|_2^2) \\
=& Pr(f(z_0\omega_d^j) > |p(z_0\omega_d^j)|^2 - \tilde{m}d\|p\|_2^2) \\
\le& Pr(f(z_0\omega_d^j) > \beta^2\|p\|_2^2 - \tilde{m}d\|p\|_2^2) \\
\le& Pr\left(f(z_0\omega_d^j) > \frac{1}{L^2}\sqrt{\frac{(2d-1)\sigma^2}{2N-1}W((2N-1)^2)}\|p\|_2^2\right) \\
\le& Pr\left(f(z_0\omega_d^j) > \sqrt{\frac{(2d-1)\sigma^2}{2N-1}W((2N-1)^2)}\right) \\
\le& \frac{e^{-\frac{1}{2}W((2N-1)^2)}}{\sqrt{W((2N-1)^2)2\pi}} \\
=& \frac{1}{\sqrt{e^{W((2N-1)^2)}W((2N-1)^2)2\pi}} \\
=& \frac{1}{(2N-1)\sqrt{2\pi}}
\end{aligned}$$

□

With the failure probability given above, we may now write the mean squared error as a sum of the error in the fail condition and the mean square error conditioned on the lower bound.

**Theorem 2.7.4.** *Let* $r = \sin(\frac{2\pi}{(d-1)d^2})$, $\beta = \frac{r^{\frac{(d-1)d}{2}}\left(\frac{d-1}{2d}\right)^{d}\frac{2}{d-1}}{\left(\prod_{k=1}^{d-1}(r^k+1)\right)}$, $L > 0$, *and* $\sigma > 0$ *with* $\sigma < \frac{\sqrt[4]{e}\beta^2 L^2}{\sqrt{2d-1}}$. *Let* $\tilde{m} = \frac{1}{d}\left(\beta^2 - \frac{1}{L^2}\sqrt{\frac{(2d-1)\sigma^2}{2N-1}W((2N-1)^2)}\right)$, *where* $W$ *is the Lambert* $W$ *function, which is the inverse of the function* $g(x) = xe^x$. *For any analytic polynomial* $p \in \mathcal{P}_d$ *with* $\|p\|_2 \geq L$, *and any noise vectors* $\eta_0, \eta_1, \eta_2 \in \mathbb{R}^{2N-1}$ *composed of i.i.d. Gaussian entries with variance* $\sigma^2$, *an approximation* $\tilde{p} \in \mathcal{P}_d$ *can be obtained by the algorithm given in section 2.3 with the phaseless measurements*

$$\begin{aligned} \{|p|^2(\omega_{2N-1}^k) + (\eta_0)_k\}_{k=0}^{2N-2} \\ \{|(I - R_{0,\omega_d})p|^2(\omega_{2N-1}^k) + (\eta_1)_k\}_{k=0}^{2N-2} \\ \{|(I - iR_{0,\omega_d})p|^2(\omega_{2N-1}^k) + (\eta_2)_k\}_{k=0}^{2N-2} \end{aligned}$$

*such that if* $J(k)$, $B_1(k)$, *and* $B_2(k)$ *are as defined in lemma 2.7.1, and we define* $B_3$ *as* $B_3 = \sum_{k=0}^{2d-2} \frac{1}{\tilde{m}^{k+2}} d^{\frac{k}{2}} B_2(k)$, *then for some* $c_0 \in \mathbb{T}$

$$\mathbb{E}[\|\tilde{p} - c_0 p\|_2^2] \leq \|p\|_2^2 \frac{d}{(2N-1)\sqrt{2\pi}} + 2B_3 \frac{6d-3}{2N-1}\frac{\sigma^2}{\|p\|_2^2} + O\left(\left(\frac{\sigma}{\sqrt{2N-1}}\right)^3\right).$$

*Proof.* Define $f \in \mathcal{T}_d$ by $f = \sum_{k=1}^{2N-1} \eta_{0,k} \frac{1}{2N-1} D_{\omega_{2N-1}^k, d-1}$ and let $F$ denote the event that there does not exist a $z$ on the unit circle such that $\min\{|p(z\omega_d^j)|^2 + f(z\omega_d^j)\}_{j=0}^{d-1} \geq \tilde{m}d\|p\|_2^2$ and $|p(z\omega_d)|^2 \geq \tilde{m}d\|p\|_2^2$. Then $Pr(F^c) \leq 1$ and by Theorem 2.7.3 $Pr(F) \leq \frac{d}{(2N-1)\sqrt{2\pi}}$. If event $F$ occurs, let $\tilde{p} = 0$. Otherwise $\tilde{p}$ is constructed using the algorithm, and by Theorem

2.7.2

$$\begin{aligned}\mathbb{E}\left[\|\tilde{p} - c_0 p\|_2^2 | F^c\right] \leq & 2\left(\sum_{k=0}^{2d-2} \frac{1}{\tilde{m}^{k+2}} d^{\frac{k}{2}} B_2(k)\right) \frac{6d-3}{2N-1} \frac{\sigma^2}{\|p\|_2^2} + O\left(\left(\frac{\sigma}{\sqrt{2N-1}}\right)^3\right) \\ = & 2B_3 \frac{6d-3}{2N-1} \frac{\sigma^2}{\|p\|_2^2} + O\left(\left(\frac{\sigma}{\sqrt{2N-1}}\right)^3\right).\end{aligned}$$

Thus, by the law of total expectation

$$\begin{aligned}\mathbb{E}[\|\tilde{p} - c_0 p\|_2^2] \leq & \mathbb{E}\left[\|\tilde{p} - c_0 p\|_2^2 | F\right] Pr(F) + \mathbb{E}\left[\|\tilde{p} - c_0 p\|_2^2 | F^c\right] Pr(F^c) \\ \leq & \|p\|_2^2 \frac{d}{(2N-1)\sqrt{2\pi}} + 2B_3 \frac{6d-3}{2N-1} \frac{\sigma^2}{\|p\|_2^2} + O\left(\left(\frac{\sigma}{\sqrt{2N-1}}\right)^3\right).\end{aligned}$$

□

# Chapter 3

# Recovery from randomized noisy measurements

## 3.1 Randomized polynomials

There are many benefits to considering random polynomials rather than fixed polynomials. Among these is the fact that the worst-case polynomial is now a rare occurrence, and so we may make probabilistic statements with better error bounds. To get a random polynomial from a fixed polynomial, we will consider the polynomial $\Phi x$, where $\Phi$ is a random matrix. If $\Phi$ is invertible, then we gain all of the error benefits of a random polynomial while still having the ability to recover a fixed polynomial. If $x$ is a sparse vector, then we can recover $x$ even when $\Phi$ has much fewer rows than columns.

To gain all of these benefits, we need to know what distribution $\Phi x$ will have.

**Proposition 3.1.1.** *Fix $x \in \mathbb{C}^M$. If $\Phi$ is a $d \times M$ complex random matrix with entries*

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*whose real and imaginary parts are drawn independently at random from a normal distribution with mean* $0$ *and variance* $\frac{1}{2d}$*, then* $\Phi x \in \mathbb{C}^d$ *has entries whose real and imaginary parts are drawn independently at random from a normal distribution with mean* $0$ *and variance* $\frac{\|x\|_2}{2d}$.

The following corollary is a special case of the above proposition.

**Corollary 3.1.2.** *Fix* $x \in \mathbb{C}^M$*. If* $\Phi$ *is a* $d \times M$ *complex random matrix with entries whose real and imaginary parts are drawn independently at random from a normal distribution with mean* $0$ *and variance* $\frac{1}{2d}$*, and if* $p \in \mathcal{P}_d$ *is the polynomial whose coefficients are equal to* $\Phi x$*, then for any* $z$ *on the unit circle, the values* $\frac{1}{\sqrt{d}} p(z\omega_d^j)$ *have real and imaginary parts that are drawn independently at random from a normal distribution with mean* $0$ *and variance* $\frac{\|x\|_2}{2d}$*, by a change of basis.*

With this knowledge of the distribution of our random vector, we may obtain a probabilistic lower bound on the coefficients with respect to our chosen basis.

**Proposition 3.1.3.** *Let* $c \geq 0$*. For any* $y \in \mathbb{C}^d$ *with entries whose real and imaginary parts are drawn independently at random from a normal distribution with mean* $0$ *and variance* $\frac{c}{2d}$*, the probability that any one value* $|y_j|$ *is greater than* $\frac{c}{d^2}$ *is* $e^{\frac{-2}{d^2}}$ *and the probability that all values* $|y_j|$ *are greater than* $\frac{c}{d^2}$ *is* $e^{\frac{-2}{d}}$*.*

*Proof.* Note that for any $j$,

$$Pr(|y_j| \geq \tfrac{c}{d^2}) = Pr(|y_j|^2 \geq \tfrac{c^2}{d^4})$$

which is the tail probability of a Chi squared distribution with 2 degrees of freedom. Then

$$Pr(|y_j|^2 \geq \tfrac{c^2}{d^4}) = Pr\left(\frac{|y_j|^2}{\left(\frac{c}{2d}\right)^2} \geq \frac{4}{d^2}\right) = e^{\frac{-2}{d^2}}$$

Then the probability all values satisfy this bound is $\left(e^{\frac{-2}{d^2}}\right)^d = e^{\frac{-2}{d}}$. $\square$

These bounds allow reconstruction of a noisy signal without using polynomial interpolation, but with a small probability of failure. In fact, this allows reconstruction using fewer measurements than the theoretical minimum needed for perfect reconstruction.

**Theorem 3.1.4.** *Fix $x \in \mathbb{C}^d$. Let $\Phi$ be an invertible $d \times d$ complex random matrix with entries whose real and imaginary parts are drawn independently at random from a normal distribution with mean 0 and variance $\frac{1}{2d}$. For any $\alpha \in (0,1)$ and any $\epsilon \in \mathbb{R}^{3d-2}$ with $\|\epsilon\|_\infty \leq \frac{\alpha}{d^4}\|x\|_2^2$, define $C \in \mathbb{R}$ such that $C = \frac{(1+\sqrt{2})\|\epsilon\|_\infty + \|\Phi x\|_\infty^2}{\frac{1-\alpha}{d^4}\|\Phi x\|_\infty^2}$. If $w$ is as defined in lemma 2.4.3 then with probability $e^{\frac{-2}{d}}$, a vector $y$ may be obtained such that*

$$\left\| y - \frac{\overline{\langle \Phi x, e_1 \rangle}}{|\langle \Phi x, e_1 \rangle|} x \right\|_2 \leq \|\Phi^{-1}\|_2 \|w(C, \tfrac{1-\alpha}{d^4})\|_2 \frac{\|\epsilon\|_2}{\|\Phi x\|_\infty}$$

*by using the generalized step 3 algorithm presented in section 2.4 with the phaseless measurements*

$$\begin{aligned} \{|\langle \Phi x, e_k \rangle|^2 + \epsilon_k\}_{k=0}^{d-1} \\ \{|\langle \Phi x, e_k - e_{k+1} \rangle|^2 + \epsilon_{k+d}\}_{k=0}^{d-2} \\ \{|\langle \Phi x, e_k - ie_{k+1} \rangle|^2 + \epsilon_{k+2d-1}\}_{k=0}^{d-2}. \end{aligned}$$

*Proof.* By Proposition 3.1.1, $\Phi x \in \mathbb{C}^d$ has entries whose real and imaginary parts are drawn independently at random from a normal distribution with mean 0 and variance $\frac{\|x\|_2}{2d}$. Then by Proposition 3.1.3, $\min_j\{|\langle \Phi x, e_j \rangle|\} \geq \frac{\|x\|_2}{d^2}$ with probability $e^{\frac{-2}{d}}$. If this holds, then for all $j$ from 1 to $d$, $|\langle \Phi x, e_j \rangle|^2 + \epsilon_j \geq (\frac{1}{d^4} - \frac{\alpha}{d^4})\|x\|_2^2 \geq \frac{1-\alpha}{d^4}\|x\|_\infty^2$. Thus, by Lemma 2.4.3, a vector $\tilde{y}$ may be obtained such that

$$\sum_{k=1}^{d}\left|\tilde{y}_k - \frac{\overline{\langle \Phi x, e_1\rangle}}{|\langle \Phi x, e_1\rangle|}\langle \Phi x, e_k\rangle\right| \leq \|w(C, \tfrac{1-\alpha}{d^4})\|_2 \frac{\|\epsilon\|_2}{\|\Phi x\|_\infty}.$$

Thus

$$\begin{aligned}\left\|\Phi^{-1}\tilde{y} - \frac{\overline{\langle \Phi x, e_1\rangle}}{|\langle \Phi x, e_1\rangle|}x\right\|_2 \leq& \|\Phi^{-1}\|_2 \left\|\tilde{y} - \frac{\overline{\langle \Phi x, e_1\rangle}}{|\langle \Phi x, e_1\rangle|}\Phi x\right\|_2 \\ \leq& \|\Phi^{-1}\|_2 \left\|\tilde{y} - \frac{\overline{\langle \Phi x, e_1\rangle}}{|\langle \Phi x, e_1\rangle|}\Phi x\right\|_1 \\ \leq& \|\Phi^{-1}\|_2 \|w(C, \tfrac{1-\alpha}{d^4})\|_2 \frac{\|\epsilon\|_2}{\|\Phi x\|_\infty}.\end{aligned}$$

□

## 3.2 Compressive sensing

We say a vector $x$ is $s$-sparse if $x$ has only $s$ or fewer nonzero entries, and we say a vector $x$ is nearly $s$-sparse if there exists an $s$-sparse vector that is a small $l^1$ distance away from $x$. For any vector $x$, we define $\|x\|_0$ to be equal to the number of nonzero entries of $x$, which is the smallest number $s$ such that $x$ is $s$-sparse. For any vector $x \in \mathbb{C}^N$, we define the error of best $s$-term approximation to $x$ by

$$\sigma_s(x)_1 = \min_{z\in\mathbb{C}^N, \|z\|_0\leq s} \|x - z\|_1$$

where a best $s$-term approximation to $x$ is given by

$$H_s(x) = \arg\min_{z\in\mathbb{C}^N, \|z\|_0\leq s} \|x - z\|_1.$$

Note that this best $s$-term approximation is not necessarily unique for a given $x$, but that the error $\sigma_s(x)$ and the norm $\|H_s(x)\|_2$ are independent of the choice of $H_s(x)$.

The idea behind compressive sensing is that a system of measurements that would be underdetermined for recovery of an arbitrary vector can be sufficient to recover a sparse or nearly sparse vector to a high degree of accuracy. This is usually established using the restricted isometry property or the more general robust null space property[40].

**Definition 3.2.1.** For a real or complex $d \times M$ matrix $\Phi$ and a positive integer $s \leq M$, we say that $\Phi$ satisfies the $s$-restricted isometry property with isometry constant $\delta_s$ if for each $s$-sparse vector $x \in \mathbb{R}^N$ or $x \in \mathbb{C}^N$, respectively, we have

$$(1-\delta_s)\|x\|_2^2 \leq \|\Phi x\|_2^2 \leq (1+\delta_s)\|x\|_2^2 .$$

**Definition 3.2.2.** For a matrix $\Phi \in \mathbb{C}^{d\times M}$ and a positive integer $s \leq M$, we say that $\Phi$ satisfies the $\ell_2$-robust null space property of order $s$ with constants $0 < \rho < 1$ and $\tau > 0$ if, for each set $S \subset \{1, \ldots, M\}$ with $|S| \leq s$, and each $x \in \mathbb{C}^M$, we have

$$\|x_S\|_2 \leq \frac{\rho}{\sqrt{s}}\|x_{S^c}\|_1 + \tau\|\Phi x\|_2$$

or equivalently,

$$\|H_s(x)\|_2 \leq \frac{\rho}{\sqrt{s}}\sigma_s(x)_1 + \tau\|\Phi x\|_2 .$$

It has been shown that the appropriate restricted isometry property implies the robust null space property.

**Theorem 3.2.3** (Theorem 6.13 in [40]). *Suppose that $\Phi \in \mathbb{C}^{d\times M}$ satisfies the $2s$-restricted isometry property with isometry constant $\delta_{2s} < \frac{4}{\sqrt{41}}$. Then $\Phi$ satisfies the $\ell_2$-robust null* *space property of order s with constants*

$$\rho = \frac{\delta_{2s}}{\sqrt{1-\delta_{2s}^2} - \delta_{2s}/4}$$

*and*

$$\tau = \frac{\sqrt{1+\delta_{2s}}}{\sqrt{1-\delta_{2s}^2} - \delta_{2s}/4}.$$

It has also been shown that for any matrix $\Phi$ satisfying the $\ell_2$-robust null space property, minimizing the $\ell_1$ norm of a vector $x$ subject to constraints involving $\Phi x$ can recover any $s$-sparse vector.

**Theorem 3.2.4** (Theorem 4.22 in [40])**.** *Suppose that $\Phi \in \mathbb{C}^{d\times M}$ satisfies the $\ell_2$-robust null space property of order $s$ with constants $0 < \rho < 1$ and $\tau > 0$. Then, for any $x \in \mathbb{C}^M$ and $y \in \mathbb{C}^d$ satisfying $\|y - \Phi x\|_2 \leq \eta$, the solution $x^{\#}$ to*

$$\arg\min_{\tilde{x}\in\mathbb{C}^M} \|\tilde{x}\|_1 \qquad \textit{subject to} \qquad \|y - \Phi\tilde{x}\|_2 \leq \eta$$

*satisfies*

$$\|x - x^{\#}\|_2 \leq \frac{(1+\rho)^2}{(1-\rho)\sqrt{s}} \left( \inf_{z\in\mathbb{C}^N, \|z\|_0 \leq s} \|x - z\|_1 \right) + \frac{(3+\rho)\tau}{1-\rho}\eta$$

## 3.3 Compressive phase retrieval

In this section we combine the oversampling results for the phase retrieval algorithm with the compressive sensing result above as in [45] to obtain a combined algorithm with a linear error bound. If it is known that the signal is sparse, then this combined algorithm may be used with undersampled measurements compared to what would be needed in the non-compressive case. The oversampling results mean that if there are more measurements than this needed (undersampled) amount of measurements, the accuracy of the combined algorithm is improved.

To combine these results, we first need a matrix that satisfies the $\ell_2$-robust null space property. By 3.2.3 a matrix satisfying the appropriate restricted isometry property will suffice. Using exercises from [40] it is shown that complex Gaussian matrixes satisfy the restricted isometry property with high probability.

**Proposition 3.3.1** (Exercise 9.5 in [40])**.** *If $\Phi$ is a $d \times s$ complex random matrix with $d > s$ and entries whose real and imaginary parts are drawn independently at random from a normal distribution with mean 0 and variance 1, then the maximal and minimal singular values $\sigma_{max}$ and $\sigma_{min}$ of $\Phi/\sqrt{2d}$ are for $t > 0$ contained in the interval $[1-\sqrt{\frac{s}{d}}-t, 1+\sqrt{\frac{s}{d}}+t]$ with a probability of*

$$\mathbb{P}\left(1 - \sqrt{\tfrac{s}{d}} - t \leq \sigma_{min}, \sigma_{max} \leq 1 + \sqrt{\tfrac{s}{d}} + t\right) \geq 1 - 2e^{-dt^2}.$$

*The proof of this is similar to the real case* [40, Theorem 9.26]*, by identifying the space $\mathbb{C}^{d\times s}$ with the space $\mathbb{R}^{2ds}$ instead of $\mathbb{R}^{ds}$.*

Using a union bound as in the proof of [40, Theorem 9.27], we then get that the restricted isometry constant $\delta_s$ of $\Phi/\sqrt{2d}$ is bounded by

$$\mathbb{P}\left(\delta_s > 2\left(\sqrt{\tfrac{s}{d}} + t\right) + \left(\sqrt{\tfrac{s}{d}} + t\right)^2\right) \leq 2\left(\frac{eM}{s}\right)^s e^{-dt^2}.$$

Combining these results from [40], we achieve the appropriate restricted isometry property.

**Proposition 3.3.2.** *A complex random matrix $\Phi$ with entries whose real and imaginary parts are drawn independently at random from a normal distribution with mean 0 and variance $1/(2d)$ achieves an RIP constant $\delta_{2s} < 4/\sqrt{41}$ with probability $1 - 2\left(\frac{eM}{2s}\right)^{2s} e^{-dt^2}$*

*if there exists*

$$t > \sqrt{\frac{2s}{d} \ln\left(\frac{eM}{2s}\right)}$$

*such that*

$$2(\sqrt{2s/d} + t) + (\sqrt{2s/d} + t)^2 < \frac{4}{\sqrt{41}}.$$

Note that if we solve the above equations for $d$, we get that such a $t$ exists if and only if

$$d > \frac{2s\left(\sqrt{\ln(\frac{eM}{2s})} + 1\right)^2}{\left(\sqrt{1 + \frac{4}{\sqrt{41}}} - 1\right)^2}.$$

Thus, for any fixed $M$ and $s$, if $d$ is sufficiently large, then a random Gaussian $d \times M$ matrix $\Phi$ will satisfy the conditions of Theorem 3.2.4 with high probability. We let $B$ be the matrix associated with the linear measurements

$$\{p(\omega_{2N-1}^k)\}_{k=0}^{2N-2}$$

$$\{(I - R_{0,\omega_d})p(\omega_{2N-1}^k)\}_{k=0}^{2N-2}$$

$$\{(I - iR_{0,\omega_d})p(\omega_{2N-1}^k)\}_{k=0}^{2N-2}$$

and consider measurements of a vector $x$ of the form $|B\Phi x|^2 + \epsilon$, where $\epsilon \in \mathbb{R}^{6m-3}$ is a noise vector and $|\bullet|^2$ is the squared modulus taken component-wise. This gives the framework that we use for compressive phase retrieval.

**Theorem 3.3.3.** *Let $N \geq d \geq s$. Let $x \in \mathbb{C}^M$, let $\eta_0, \eta_1, \eta_2 \in \mathbb{R}^{2N-1}$, and let $\Phi \in \mathbb{C}^{d \times M}$ satisfy the $\ell_2$-robust null space property of order $s$ with constants $0 < \rho < 1$ and $\tau > 0$. If $\omega_{2N-1} = e^{\frac{2i\pi}{2N-1}}$ and $\omega_d = e^{\frac{2i\pi}{d}}$, then for $j$ from $0$ to $6d-4$ and $k$ from $0$ to $d-1$ let*

*$B \in \mathbb{C}^{(6N-3)\times d}$ be given by*

$$B_{j,k} = \begin{cases} \omega_{2N-1}^{jk} & \text{if } 0 \le j < 2N-1, \\ \omega_{2N-1}^{jk} - (\omega_{2N-1}^{j}\omega_d)^k & \text{if } 2N-1 \le j < 4N-2, \\ \omega_{2N-1}^{jk} - i(\omega_{2N-1}^{j}\omega_d)^k & \text{if } 4N-2 \le j < 6N-3 \end{cases}$$

*and let $\epsilon \in \mathbb{R}^{6N-3}$ be given by*

$$\epsilon_j = \begin{cases} (\eta_0)_j & \text{if } 0 \le j < 2N-1, \\ (\eta_1)_{j-(2N-1)} & \text{if } 2N-1 \le j < 4N-2, \\ (\eta_2)_{j-(4N-2)} & \text{if } 4N-2 \le j < 6N-3\,. \end{cases}$$

*Let $r = \sin(\frac{2\pi}{(d-1)d^2})$, $0 < \alpha < 1$, and $\beta = \frac{r^{\frac{(d-1)d}{2}}\left(\frac{d-1}{2d}\right)^d \frac{2}{d-1}}{\left(\prod_{k=1}^{d-1}(r^k+1)\right)}$. If $\|\eta_0\|_2 \le \frac{\sqrt{2N-1}}{\sqrt{2d-1}}\alpha\beta^2\|\Phi x\|_2^2$ and $x$ satisfies the approximate sparsity requirement*

$$\sigma_s(x)_1 < \frac{\sqrt{s}}{\rho}\|H_s(x)\|_2$$

*then an approximation $x^{\#}$ for $x$ can be reconstructed from the vector $|B\Phi x|^2 + \epsilon$ (where $|\bullet|^2$ is taken component-wise), such that*

$$\|c_0 x - x^{\#}\|_2 \le \frac{C_1}{\sqrt{s}}\sigma_s(x)_1 + \frac{C_2}{\sqrt{2N-1}}\frac{\|\epsilon\|_2}{\|H_s(x)\|_2 - \frac{\rho}{\sqrt{s}}\sigma_s(x)_1}$$

*where*

*$C_1 = \frac{(1+\rho)^2}{1-\rho}$, and $C_2 = \frac{(3+\rho)\tau^2}{1-\rho}\left\|w\left(C, \frac{1}{d}\beta^2(1-\alpha)\right)\right\|_2\sqrt{2}$ with $w$ and $C$ from Theorem 2.6.3, and $c_0 \in \mathbb{C}$, with $|c_0| = 1$.*

*Proof.* Consider the polynomial $p_{\Phi x} \in \mathcal{P}_d$ defined such that $p_{\Phi x}(z) = \sum_{k=1}^{d}\langle x, \phi_k^*\rangle z^{k-1}$, where $\phi_k$ is the $k$-th row of $\Phi$. This polynomial has monomial coefficients that are precisely equal to the coefficients of the vector $\Phi x$. Thus, the measurements $\left(|BAx|^2+\epsilon\right)_j$ are precisely the measurements needed to apply the 3-step phase retrieval algorithm to recover $p_{\Phi x}$. The recovered vector will have an error as specified by Theorem 2.6.3. Using Theorem 2.6.3, for some $c_0$ on the unit circle, we obtain a polynomial $\tilde{p}\in\mathcal{P}_d$ satisfying

$$\|\tilde{p}-c_0p_{\Phi x}\|_2 \le \left\|w\left(C,\frac{1}{d}\beta^2(1-\alpha)\right)\right\|_2 \frac{\sqrt{2}}{\sqrt{2N-1}} \frac{\sqrt{\|\eta_0\|_2^2+\|\eta_1\|_2^2+\|\eta_2\|_2^2}}{\|p\|_2}.$$

where $C$ and $w$ are as in Theorem 2.6.3. If $y$ is the vector of monomial coefficients of $\tilde{p}$, then

$$\|y-c_0\Phi x\|_2=\|\tilde{p}-c_0p_{\Phi x}\|_2 \le \left\|w\left(C,\frac{1}{d}\beta^2(1-\alpha)\right)\right\|_2 \frac{\sqrt{2}}{\sqrt{2N-1}} \frac{\|\epsilon\|_2}{\|\Phi x\|_2}.$$

Using this, we may apply Theorem 3.2.4 to show that the solution $x^{\#}$ to

$$\arg\min_{\tilde{x}\in\mathbb{C}^N}\|\tilde{x}\|_1 \qquad \text{subject to} \qquad \|y-c_0A\tilde{x}\|_2 \le \left\|w\left(C,\frac{1}{d}\beta^2(1-\alpha)\right)\right\|_2 \frac{\sqrt{2}}{\sqrt{2N-1}} \frac{\|\epsilon\|_2}{\|\Phi x\|_2}$$

satisfies the random bound

$$\begin{aligned}\|c_0x-x^{\#}\|_2 &\le \frac{(1+\rho)^2}{(1-\rho)\sqrt{s}}\sigma_s(x)_1+\frac{(3+\rho)\tau}{1-\rho}\left\|w\left(C,\tfrac{1}{d}\beta^2(1-\alpha)\right)\right\|_2 \frac{\sqrt{2}}{\sqrt{2N-1}} \frac{\|\epsilon\|_2}{\|\Phi x\|_2}\\ &= \frac{C_1}{\sqrt{s}}\sigma_s(x)_1+\frac{C_2}{\tau\sqrt{2N-1}}\frac{\|\epsilon\|_2}{\|\Phi x\|_2}.\end{aligned}$$

To eliminate the random term $\|\Phi x\|_2$ in the denominator, we use the robust null space property to get

$$\|\Phi x\|_2 \ge \frac{1}{\tau}\left(\|H_s(x)\|_2-\frac{\rho}{\sqrt{s}}\sigma_s(x)_1\right).$$

Thus,

$$\begin{aligned}\|c_0 x - x^{\#}\|_2 \leq & \frac{C_1}{\sqrt{s}}\sigma_s(x)_1 + \frac{C_2}{\tau\sqrt{2N-1}}\frac{\|\epsilon\|_2}{\|\Phi x\|_2} \\ \leq & \frac{C_1}{\sqrt{s}}\sigma_s(x)_1 + \frac{C_2}{\sqrt{2N-1}}\frac{\|\epsilon\|_2}{\|H_s(x)\|_2 - \frac{\rho}{\sqrt{s}}\sigma_s(x)_1}.\end{aligned}$$

□

If $x$ is $s$-sparse, then the error bound simplifies to a linear bound in the noise-to-signal ratio. An example of this can be seen in figure 3.1. In this experiment, a single sparse vector was randomly generated, and then 100 matrices were randomly generated, and 100 noise vectors with varying norms were randomly generated. The results show a linear relationship between the input noise and the recovery error.

**Corollary 3.3.4.** *If the assumptions of the theorem 3.3.3 hold and $x$ is $s$-sparse, then the recovery algorithm results in $x^{\#}$ such that*

$$\min_{|c|=1} \|cx - x^{\#}\|_2 \leq \frac{C_2}{\sqrt{2N-1}}\frac{\|\epsilon\|_2}{\|x\|_2}.$$

Together with the random selection of normally, independently distributed entries as in Proposition 3.3.2, we achieve overwhelming probability of approximate recovery.

**Corollary 3.3.5.** *If $\Phi \in \mathbb{C}^{d\times M}$ is a complex random matrix with entries whose real and imaginary parts are drawn independently at random from a normal distribution with mean $0$ and variance $1/(2d)$, with $s$, $d$, $M$ and $t > 0$ chosen according to the assumption of Proposition 3.3.2, then the error bound in the theorem 3.3.3 holds for each $x \in \mathbb{C}^M$ with a probability bounded below by $1 - 2(\frac{eM}{2s})^{2s}e^{-dt^2}$.*

If we are willing to accept nonuniform recovery, then the results from proposition 3.1.3 give a smaller coefficient for the linear bound on the error.

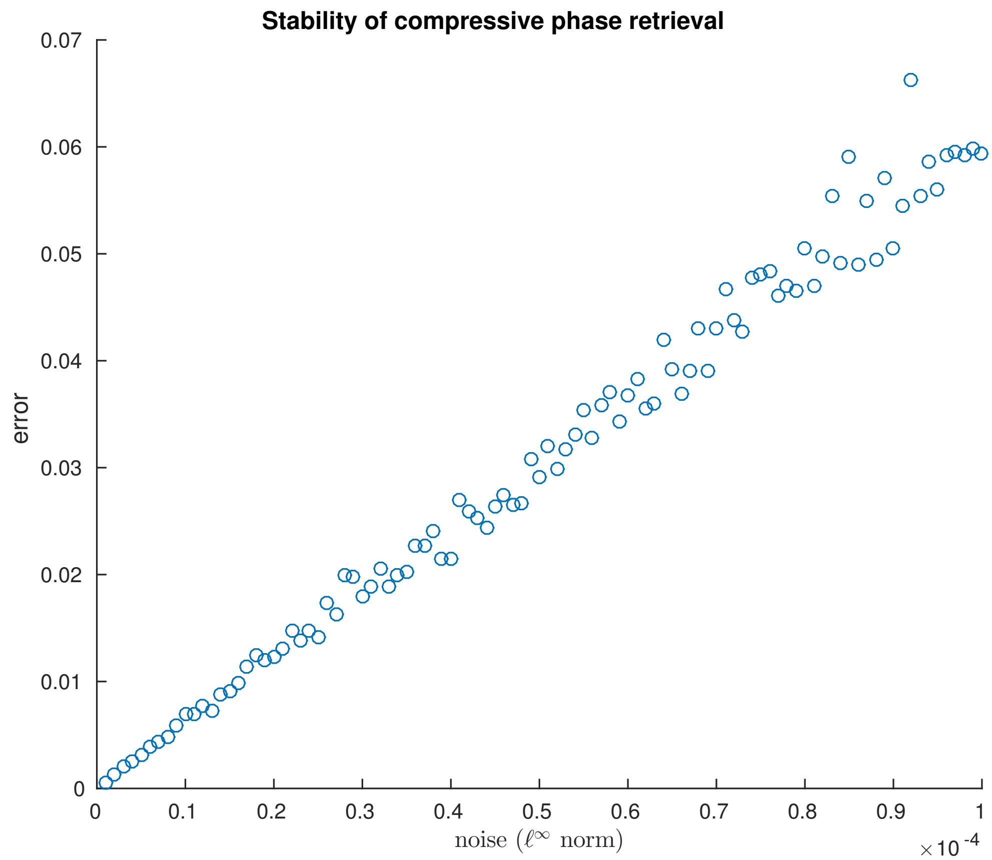


Figure 3.1: The $L^2$ distance between the signal recovered by the compressive phase retrieval algorithm and the actual signal for a 20-sparse vector in dimension $M = 512$ with 120 measurements as a function of the maximal noise magnitude.

**Theorem 3.3.6.** *Given $x \in \mathbb{C}^M$ and $\eta_0, \eta_1, \eta_2 \in \mathbb{R}^{2N-1}$, let $\Phi \in \mathbb{C}^{d\times M}$ be a complex random matrix with entries whose real and imaginary parts are drawn independently at random from a normal distribution with mean 0 and variance $1/(2d)$, with $s$, $d$, $M$ and $t > 0$ chosen according to the assumption of Proposition 3.3.2. If $\omega_{2N-1} = e^{\frac{2i\pi}{2N-1}}$ and $\omega_d = e^{\frac{2i\pi}{d}}$, then for $j$ from 0 to $6d-4$ and $k$ from 0 to $d-1$ let $B \in \mathbb{C}^{(6N-3)\times d}$ be given by*

$$B_{j,k} = \begin{cases} \omega_{2N-1}^{jk} & \text{if } 0 \le j < 2N-1, \\ \omega_{2N-1}^{jk} - (\omega_{2N-1}^{j}\omega_d)^k & \text{if } 2N-1 \le j < 4N-2, \\ \omega_{2N-1}^{jk} - i(\omega_{2N-1}^{j}\omega_d)^k & \text{if } 4N-2 \le j < 6N-3 \end{cases}$$

*and let $\epsilon \in \mathbb{R}^{6N-3}$ be given by*

$$\epsilon_j = \begin{cases} (\eta_0)_j & \text{if } 0 \le j < 2N-1, \\ (\eta_1)_{j-(2N-1)} & \text{if } 2N-1 \le j < 4N-2, \\ (\eta_2)_{j-(4N-2)} & \text{if } 4N-2 \le j < 6N-3\,. \end{cases}$$

*Let $r = \sin(\frac{2\pi}{(d-1)d^2})$ and $0 < \alpha < 1$. If $\|\eta_0\|_2 \le \frac{\sqrt{2N-1}}{\sqrt{2d-1}}\frac{\alpha}{d^2}\|\Phi x\|_2^2$ and $x$ satisfies the approximate sparsity requirement*

$$\sigma_s(x)_1 < \frac{\sqrt{s}}{\rho}\|H_s(x)\|_2$$

*then with probability bounded below by $e^{-\frac{2}{d}} - 2(\frac{eM}{2s})^{2s}e^{-dt^2}$, an approximation $x^\#$ for $x$ can be reconstructed from the vector $|B\Phi x|^2 + \epsilon$ (where $|\bullet|^2$ is taken component-wise), such that*

$$\|c_0 x - x^\#\|_2 \le \frac{C_1}{\sqrt{s}}\sigma_s(x)_1 + \frac{C_2}{\sqrt{2N-1}}\frac{\|\epsilon\|_2}{\|H_s(x)\|_2 - \frac{\rho}{\sqrt{s}}\sigma_s(x)_1}$$

*where*

*$C_1 = \frac{(1+\rho)^2}{1-\rho}$, and $C_2 = \frac{(3+\rho)\tau^2}{1-\rho} \left\| w\left(C, \frac{1}{d^3}(1-\alpha)\right)\right\|_2 \sqrt{2}$ with $w$ and $C$ from Theorem 2.6.3, and $c_0 \in \mathbb{C}$, with $|c_0| = 1$.*

*Proof.* First, we note that $\Phi$ satisfies the $\ell_2$-robust null space property of order $s$ with constants $0 < \rho < 1$ and $\tau > 0$ with probability bounded below by $1 - 2(\frac{eM}{2s})^{2s} e^{-dt^2}$. In this case $\rho = \frac{\delta_{2s}}{\sqrt{1-\delta_{2s}^2} - \delta_{2s}/4}$ and $\tau = \frac{\sqrt{1+\delta_{2s}}}{\sqrt{1-\delta_{2s}^2} - \delta_{2s}/4}$ for any choice of $\delta_{2s} < 4/\sqrt{41}$.

Consider the polynomial $p_{\Phi x} \in \mathcal{P}_d$ defined such that $p_{\Phi x}(z) = \sum_{k=1}^{d} \langle x, \phi_k^* \rangle z^{k-1}$, where $\phi_k$ is the $k$-th row of $\Phi$. This polynomial has monomial coefficients that are precisely equal to the coefficients of the vector $\Phi x$. Thus, the measurements $\left(|BAx|^2 + \epsilon\right)_j$ are precisely the measurements needed to apply the 3-step phase retrieval algorithm to recover $p_{\Phi x}$. By corollary 3.1.2 and proposition 3.1.3, for any choice of $z_0 \in \mathbb{T}$, all values $\frac{1}{\sqrt{d}} p(z_0 \omega_d^j)$ are greater than $\frac{\|x\|_2}{d^2}$ with probability $e^{-\frac{2}{d}}$. Thus,

$$\min \left\{ |p(\omega_d^j z_0)|^2 - \frac{\sqrt{2d-1}}{\sqrt{2N-1}} \|\eta_0\|_1 \right\}_{j=0}^{d-1} \geq \frac{1}{d^2} \|p\|_2^2 - \frac{\sqrt{2d-1}}{\sqrt{2N-1}} \|\eta_0\|_1 \geq \frac{1}{d^2}(1-\alpha)\|p\|_2^2$$

for all $j$ from 0 to $d-1$ and we may use $z_0$ and $\tilde{m} = \frac{1}{d^3}(1-\alpha)$ in theorem 2.6.2. When we apply the theorem, we get $\tilde{p} \in \mathcal{P}_d$ such that

$$\|\tilde{p} - c_0 p\|_2 \leq \left\| w\left(C, \frac{1}{d^3}(1-\alpha)\right)\right\|_2 \frac{\sqrt{2}}{\sqrt{2N-1}} \frac{\sqrt{\|\eta_0\|_2^2 + \|\eta_1\|_2^2 + \|\eta_2\|_2^2}}{\|p\|_2}.$$

If $y$ is the vector of monomial coefficients of $\tilde{p}$, then

$$\|y - c_0 \Phi x\|_2 = \|\tilde{p} - c_0 p_{\Phi x}\|_2 \leq \left\| w\left(C, \frac{1}{d^3}(1-\alpha)\right)\right\|_2 \frac{\sqrt{2}}{\sqrt{2N-1}} \frac{\|\epsilon\|_2}{\|\Phi x\|_2}.$$

Using this, we may apply Theorem 3.2.4 to show that the solution $x^\#$ to

$$\arg\min_{\tilde{x} \in \mathbb{C}^N} \|\tilde{x}\|_1 \qquad \text{subject to} \qquad \|y - c_0 A\tilde{x}\|_2 \leq \left\| w\left(C, \frac{1}{d^3}(1-\alpha)\right)\right\|_2 \frac{\sqrt{2}}{\sqrt{2N-1}} \frac{\|\epsilon\|_2}{\|\Phi x\|_2}$$

satisfies the random bound

$$
\begin{aligned}
\|c_0 x - x^{\#}\|_2 \leq & \frac{(1+\rho)^2}{(1-\rho)\sqrt{s}}\sigma_s(x)_1 + \frac{(3+\rho)\tau}{1-\rho}\left\|w\left(C, \tfrac{1}{d^3}(1-\alpha)\right)\right\|_2 \frac{\sqrt{2}}{\sqrt{2N-1}}\frac{\|\epsilon\|_2}{\|\Phi x\|_2} \\
= & \frac{C_1}{\sqrt{s}}\sigma_s(x)_1 + \frac{C_2}{\tau\sqrt{2N-1}}\frac{\|\epsilon\|_2}{\|\Phi x\|_2} .
\end{aligned}
$$

To eliminate the random term $\|\Phi x\|_2$ in the denominator, we use the robust null space property to get

$$
\|\Phi x\|_2 \geq \frac{1}{\tau}\left(\|H_s(x)\|_2 - \frac{\rho}{\sqrt{s}}\sigma_s(x)_1\right) .
$$

Thus,

$$
\begin{aligned}
\|c_0 x - x^{\#}\|_2 \leq & \frac{C_1}{\sqrt{s}}\sigma_s(x)_1 + \frac{C_2}{\tau\sqrt{2N-1}}\frac{\|\epsilon\|_2}{\|\Phi x\|_2} \\
\leq & \frac{C_1}{\sqrt{s}}\sigma_s(x)_1 + \frac{C_2}{\sqrt{2N-1}}\frac{\|\epsilon\|_2}{\|H_s(x)\|_2 - \frac{\rho}{\sqrt{s}}\sigma_s(x)_1} .
\end{aligned}
$$

$\square$

# Appendix A

# Appendix

## A.1 A max-min principle for magnitudes

To ensure the stability of our phase retrieval algorithm, we need a lower bound on the magnitude of any polynomial with a given degree and norm for some choice of $d$ equally spaced points on $\mathbb{T}$. Obtaining the needed lower bound on the magnitude requires a few lemmas and definitions.

**Lemma A.1.1.** *Let $(a_k)_{k=0}^{d-1} \in \mathbb{C}^d$. For any $t \in (0,1)$, there exists at least one $n$ between $0$ and $d-1$ such that $|a_n| \geq \|a\|_1 t^{d-n}(t^{-1}-1)$.*

*Proof.* By way of contradiction, let $|a_j| < \|a\|_1 t^{d-j}(t^{-1}-1)$ for all $j$ from 0 to $d-1$. Then

$$\|a\|_1 = \sum_{j=0}^{d-1} |a_j| < \sum_{j=0}^{d-1} \|a\|_1 t^{d-j}(t^{-1}-1) = \|a\|_1 (t^{-1}-1) \sum_{l=1}^{d} t^l \leq \|a\|_1$$

This is a contradiction, so the claim holds. $\square$

**Definition A.1.2.** For $n$ from 1 to $d$, recall that for any $x \in \mathbb{C}^d$, $p_x \in \mathcal{P}_d$ such that $p_x(z) = \sum_{j=0}^{d-1} x_j z^j$. We define the $n$-th truncation of $p_x$ to be the polynomial $p_{x,n} \in \mathcal{P}_n$

such that $p_{x,n}(z) = \sum_{j=0}^{n-1} x_j z^j$. Note that the $d$-th truncation of a polynomial is the polynomial itself. Also note that even if a polynomial is nonzero, it may have truncations that are zero polynomials.

To obtain a lower bound on the entries of a full vector originating from a polynomial, a proper choice of basis is needed. It must be shown that there exists a choice of basis that provides a lower bound on the distance between any basis element and any roots of any truncations of the base polynomial.

**Lemma A.1.3.** *For any polynomial $p \in \mathcal{P}_d$, there exists a $z_0$ on the unit circle such that the linear distance between any element of $\{\omega_d^j z_0\}_{j=0}^{d-1}$ and any roots of any nonzero truncations of $p$ is at least* $\sin(\frac{2\pi}{(d-1)d^2})$.

*Proof.* For any $n$ from 1 to $d$, let $N_n$ be the number of distinct roots of the $n$-th truncation of $p$ if that truncation is nonzero, and let $N_n = 0$ if the $n$-th truncation of $p$ is a zero polynomial. Then the number of distinct roots of all nonzero truncations of $p$ is

$$N \leq \sum_{n=1}^{d} N_n \leq \sum_{n=1}^{d} (n-1) = \frac{(d-1)d}{2}$$

Then for the set $S = \{\frac{w}{|w|}\omega_d^j \mid j = 1, \ldots, d$ and $w$ is a root of a nonzero truncation of $p\}$, we know $|S| = Nd \leq \frac{(d-1)d^2}{2}$. If the elements of $S$ are ordered by their angle around the unit circle, then the average angle between adjacent elements is $\frac{2\pi}{|S|} \geq \frac{4\pi}{(d-1)d^2}$, and so there is at least one pair of adjacent elements that is separated by at least this amount. Thus, if we let $z_0$ be the midpoint between these two maximally separated elements on the unit circle, then the angle between $z_0$ and any element of $S$ is at least $\frac{2\pi}{(d-1)d^2}$. Thus the linear distance between $z_0$ and any element of the set $S$ is at least $\sin(\frac{2\pi}{(d-1)d^2})$. $\square$

With a lower bound on the distance between these roots and the chosen evaluation points, a lower bound on the minimum magnitude of any component of the reduced vector can be obtained.

**Lemma A.1.4.** *Let $r \leq 1$. For any $x \in \mathbb{C}^d$, if there exists a $z_0$ on the unit circle such that the linear distance between any element of $\{\omega_d^j z_0\}_{j=0}^{d-1}$ and any zeros of any nonzero truncations of $p_x$ is at least $r$, then for all $j$ from $0$ to $d-1$*

$$|p_x(\omega_d^j z_0)| \geq \frac{r^{\frac{(d-1)d}{2}} \left(\frac{d-1}{2d}\right)^d \frac{2}{d-1}}{\left(\prod_{k=0}^{d-1}(r^k+1)\right)} \|x\|_1 \,.$$

*Proof.* Let $n_0$ be the smallest $n$ obtained by applying Lemma A.1.1 to $x$ and $t = \frac{d-1}{2d}$. Then

$$|x_{n_0}| \geq \|x\|_1 \left(\frac{d-1}{2d}\right)^{d-n_0} \left(\left(\frac{d-1}{2d}\right)^{-1} - 1\right) = \|x\|_1 \left(\frac{d-1}{2d}\right)^{d-n_0} \frac{d+1}{d-1}$$

and for all $j < n_0$

$$|x_j| < \|x\|_1 \left(\frac{d-1}{2d}\right)^{d-j} \left(\left(\frac{d-1}{2d}\right)^{-1} - 1\right) = \|x\|_1 \left(\frac{d-1}{2d}\right)^{d-j} \frac{d+1}{d-1} \,.$$

Let

$$m(n_0, n) = \frac{r^{\frac{(n-1)n}{2}} \|x\|_1 \left(\frac{d-1}{2d}\right)^{d-n_0} \frac{2}{d-1}}{\left(\prod_{k=n_0}^{n-1}(r^k+1)\right)}$$

We prove, by induction on $n$ from $n_0+1$ to $d$, that the $n$-th truncation of $p_x$ satisfies the inequality $|p_{x,n}(\omega_d^j z_0)| \geq m(n_0, n)$ for all $j$ from 1 to $d$.

For the base case $n = n_0 + 1$, we know that

$$
\begin{aligned}
|x_{n_0}| - \sum_{j=0}^{n_0-1} |x_j| \geq & \|x\|_1 \left( \left(\frac{d-1}{2d}\right)^{d-n_0} - \sum_{j=0}^{n_0-1} \left(\frac{d-1}{2d}\right)^{d-j} \right) \frac{d+1}{d-1} \\
= & \|x\|_1 \left( \left(\frac{d-1}{2d}\right)^{d-n_0} - \sum_{l=d-n_0+1}^{d} \left(\frac{d-1}{2d}\right)^{l} \right) \frac{d+1}{d-1} \\
\geq & \|x\|_1 \left( \left(\frac{d-1}{2d}\right)^{d-n_0} - \frac{\left(\frac{d-1}{2d}\right)^{d-n_0+1}}{1-\left(\frac{d-1}{2d}\right)} \right) \frac{d+1}{d-1} \\
= & \|x\|_1 \left(\frac{d-1}{2d}\right)^{d-n_0} \frac{2}{d-1}
\end{aligned}
$$

and equality only holds if $n_0 = 0$. Then by the reverse triangle inequality

$$
\begin{aligned}
|p_{x,n_0+1}(\omega_d^j z_0)| = & \left| \sum_{k=0}^{n_0-1} x_k (\omega_d^j z_0)^k \right| \\
\geq & \left| |x_{n_0}| - \sum_{j=0}^{n_0-1} |x_j| \right| \\
\geq & \|x\|_1 \left(\frac{d-1}{2d}\right)^{d-n_0} \frac{2}{d-1} \\
\geq & \frac{r^{\frac{(n_0-1)n_0}{2}} \|x\|_1 \left(\frac{d-1}{2d}\right)^{d-n_0} \frac{2}{d-1}}{\left(\prod_{k=n_0}^{n_0-1} (r^k+1)\right)} \\
= & m(n_0, n_0)
\end{aligned}
$$

For the inductive step, assume that we have proven that $|p_{x,n}(\omega_d^j z_0)| \geq m(n_0, n)$ for a chosen value of $n$, and all $j$ from 1 to $d$. Then we choose a threshold $\tau_n = \frac{m(n_0,n)}{r^n+1}$. If the leading coefficient $x_n$ of $p_{x,n+1}$ satisfies $|x_n| > \tau_n$, then $p_{x,n+1}$ is clearly a nonzero truncation of $p_x$, so by using the factored form of $p_{x,n+1}$, for all $j$ from 1 to $d$

$$
|p_{x,n+1}(\omega_d^j z_0)| \geq |x_n| r^n > \tau_n r^n = \frac{m(n_0, n) r^n}{r^n + 1}
$$

On the other hand, if the leading coefficient satisfies $|x_n| \leq \tau_n \leq m(n_0, n)$, then for all $j$ from 1 to $d$

$$|p_{x,n+1}(\omega_d^j z_0)| \geq m(n_0, n) - |c_n| \geq m(n_0, n) - \tau_n = m(n_0, n) - \frac{m(n_0, n)}{r^n + 1} = \frac{m(n_0, n) r^n}{r^n + 1}$$

Either way, for all $j$ from 1 to $d$,

$$|p_{x,n+1}(\omega_d^j z_0)| \geq \frac{m(n_0, n) r^n}{r^n + 1} = \frac{r^{\frac{(n-1)n}{2}} \|\hat{p}\|_1 \left(\frac{d-1}{2d}\right)^{d-n_0} \frac{2}{d-1} r^n}{\left(\prod_{k=n_0}^{n} (r^k + 1)\right)} = m(n_0, n+1)$$

Thus, by using the fact that the $d$-th truncation of a polynomial is the polynomial itself, for all $j$ from 1 to $d$,

$$|p_x(\omega_d^j z_0)| \geq m(n_0, d) = \frac{r^{\frac{(d-1)d}{2}} \|x\|_1 \left(\frac{d-1}{2d}\right)^{d-n_0} \frac{2}{d-1}}{\left(\prod_{k=n_0}^{d-1} (r^k + 1)\right)} \geq \frac{r^{\frac{(d-1)d}{2}} \|x\|_1 \left(\frac{d-1}{2d}\right)^{d} \frac{2}{d-1}}{\left(\prod_{k=0}^{d-1} (r^k + 1)\right)}$$

$\square$

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